INCOMPLETE ORTHOGONAL FAMILIES AND A RELATED QUESTION ON ORTHOGONAL MATRICES

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1. A QUESTION POSED BY W. KAPLAN

Wilfred Kaplan has asked the author the following question: If we are given an incomplete orthogonal family of *continuous* functions on some interval, is there necessarily a nonnull *continuous* function orthogonal to all the given functions?

In this note we shall show that the answer is negative, even when restrictions stronger than continuity are imposed on the given functions. For definiteness we shall construct our counterexample on the circle group Γ, that is, on the real numbers modulo 2π; however, our method of construction is perfectly general. (Throughout the paper, all functions and numbers are assumed to be real-valued; but our results are also valid in the complex case.)

THEOREM 1. *Corresponding to any* $f \in L^2(\Gamma)$ *of norm* 1, *there exists a complete orthonormal basis in* $L^2(\Gamma)$ *that contains* f *and whose remaining elements are trigonometric polynomials. If moreover the Fourier coefficients of* f (*with respect to the usual trigonometric system*) *are* $O(n^{-1/2})$, *then the remaining basis functions are also uniformly bounded.*

The following generalization of the first statement in Theorem 1 is also true:

THEOREM 2. *Corresponding to any finite orthonormal set of functions in* $L^2(\Gamma)$, *there exists a complete orthonormal basis in* $L^2(\Gamma)$ *that contains the given functions and whose remaining elements are trigonometric polynomials.*

These theorems are consequences of a simple theorem concerning infinite matrices. We shall say that the infinite matrix $A = \|a_{ij}\|$ $(i, j = 1, 2, \cdots)$ of real numbers is *orthogonal* if its rows form an orthonormal basis for the Hilbert space ℓ^2. If A is orthogonal, so is A^T. We denote by A_{i*} and A_{*j} the i^{th} row and j^{th} column, respectively, of A.

THEOREM 3. *Let there be given* n *orthogonal unit vectors in* ℓ^2, *which we write as row vectors*

$$A_{i*} = \{a_{ij}\} \qquad (1 \leq j < \infty, \ 1 \leq i \leq n).$$

Suppose the square matrix $\|a_{ij}\|$ $(1 \leq i, j \leq n)$ *is nonsingular. Then there exists an infinite orthogonal matrix* $A = \|a_{ij}\|$ *with the given vectors as its first* n *rows and with* $a_{ij} = 0$ *for* $j > i > n$. *These conditions uniquely determine* A. *Moreover, in the case* $n = 1$, *if* $a_{1j} = O(\sqrt{j})$, *the sums* $\Sigma_{j=1}^{\infty} |a_{ij}|$ $(i = 2, 3, \cdots)$ *are uniformly bounded.*

To see that Theorems 1 and 2 follow from this, let $\phi_n(x)$ $(n = 1, 2, \cdots)$ denote the n^{th} function in the sequence $1, 2\sqrt{2}\cos x, 2\sqrt{2}\sin x, 2\sqrt{2}\cos 2x, 2\sqrt{2}\sin 2x, \cdots$. Since $\{\phi_n(x)\}$ is an orthonormal basis for $L^2(\Gamma)$, the same is true of the system $\psi_n(x) = \Sigma_1^{\infty} a_{nj}\phi_j(x)$ whenever $\|a_{nj}\|$ is an orthogonal matrix. Thus, to prove

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