RATIONAL APPROXIMATION TO $|x|$

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The function $|x|$ plays a central role in the theory of polynomial approximation. Indeed, Lebesgue's proof of the Weierstrass approximation theorem is based solely on the fact that the single function $|x|$ can be approximated. One can even give a proof of *Jackson's* approximation theorem by simply using an appropriate polynomial approximation to $|x|$.

Quantitatively speaking, $|x|$ has "order of approximation" $1/n$. In precise language this means that (with C_1, C_2 positive absolute constants)

(1) there exists an n^{th}-degree polynomial $P(x)$ such that, throughout $[-1, 1]$,

$$\big||x| - P(x)\big| \le \frac{C_1}{n};$$

(2) there does not exist an n^{th}-degree polynomial $P(x)$ such that, throughout $[-1, 1]$,

$$\big||x| - P(x)\big| \le \frac{C_2}{n}.$$

Suppose we now turn to the question of approximation by rational functions rather than by polynomials. While it is true that in this context the function $|x|$ loses much of its special significance, it is nevertheless of some interest to determine the order of approximation to $|x|$ by n^{th}-order rational functions (the problem was actually suggested by H. S. Shapiro).

Now it is known that, in some overall sense, rational approximation is essentially no better than polynomial approximation, and this suggests the naïve guess that the order of approximation of $|x|$ by n^{th}-order rational functions is also $1/n$. The truth, however, is remarkably far from this guess. Indeed, the purpose of the present paper is to show that this order of approximation is actually $e^{-c\sqrt{n}}$.

Notation. n is an integer greater than 4, $\xi = \exp(-n^{-1/2})$, and

$$p(x) = \prod_{k=0}^{n-1} (x + \xi^k), \qquad r(x) = x\,\frac{p(x) - p(-x)}{p(x) + p(-x)}.$$

By the *order* of a rational function we mean the maximum of the degrees of its numerator and denominator. [Note that the order of $r(x)$ is n when n is even].

THEOREM (A). $\big||x| - r(x)\big| \le 3e^{-\sqrt{n}}$ *throughout* $[-1, 1]$.

(B). *There does not exist an* n^{th}-*order rational function* $R(x)$ *such that* $\big||x| - R(x)\big| \le \frac{1}{2}e^{-9\sqrt{n}}$ *throughout* $[-1, 1]$.

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