AN INTERPOLATION PROBLEM ASSOCIATED WITH THE CONTINUUM HYPOTHESIS

P. Erdös

In the Ann Arbor Problem Book, Wetzel asked (under the date December, 1962) the following question: Let $\{f_\alpha\}$ be a family of analytic functions such that for each z the set of values $f_\alpha(z)$ is countable (we shall call this property P_0). Does it then follow that the family itself is countable?

An unsigned comment points out that if "countable" is replaced with "finite" both in the hypothesis and in the conclusion, then the result follows easily. R. C. Lyndon has remarked that if "analytic" is replaced with "infinitely differentiable," one can easily give $\mathfrak{c}$ functions f_α $(1 \leq \alpha < \Omega_\mathfrak{c})$ such that, for each z, the set $\{f_\alpha(z)\}$ contains only two values.

We shall show that the answer to Wetzel's question depends on the continuum hypothesis.

THEOREM. *If* $\mathfrak{c} > \aleph_1$, *then every family* $\{f_\alpha\}$ *with property* P_0 *is denumerable. If* $\mathfrak{c} = \aleph_1$, *some family* $\{f_\alpha\}$ *with property* P_0 *has the power* $\mathfrak{c}$. (I have been informed that R. D. Dixon proved the first part of the theorem last year.)

Proof. Assume first that $\mathfrak{c} > \aleph_1$, and let $\{f_\alpha\}$ $(1 \leq \alpha < \Omega_1)$ be a family of $\aleph_1$ distinct analytic functions. We shall show that there exists a point z_0 such that all the $\aleph_1$ values $f_\alpha(z_0)$ are distinct.

For each pair (α, β) $(1 \leq \alpha < \beta < \Omega_1)$, the set $S(\alpha, \beta)$ of values z for which $f_\alpha(z) = f_\beta(z)$ is at most denumerable. Put

$$S = \bigcup_{1 \leq \alpha < \beta < \Omega_1} S(\alpha, \beta).$$

Then S has power at most $\aleph_1$, since it is the union of $\aleph_1$ countable sets. Because $\mathfrak{c} > \aleph_1$, there exists a complex number z_0 not in S, and since all the values $f_\alpha(z_0)$ $(1 \leq \alpha < \Omega_1)$ are distinct, the first part of our theorem is proved.

Next we assume that $c = \aleph_1$. Let S be any denumerable, dense set of complex numbers, and let $\{z_\alpha\}$ $(1 \leq \alpha < \Omega_1)$ be a well-ordering of the complex numbers. We shall construct a family $\{f_\beta\}$ $(1 \leq \beta < \Omega_1)$ of distinct entire functions such that $f_\beta(z_\alpha) \in S$ whenever $1 \leq \alpha < \beta < \Omega_1$. Clearly, the set $\{f_\beta(z)\}$ will be denumerable, for each z; indeed, each point z has an index, say α, in the well-ordering; if $\beta > \alpha$, then the value $f_\beta(z)$ is in the denumerable set S; and only countably many functions have an index β with $1 \leq \beta \leq \alpha$.

The construction of $\{f_\beta\}$ is by transfinite induction. Suppose that f_β has already been defined for $1 \leq \beta < \gamma < \Omega_1$. The family $\{f_\beta\}$ $(1 \leq \beta < \gamma)$ is denumerable; we reorder it into a sequence and denote its elements by g_n. The similar reordering of $\{z_\alpha\}$ $(\alpha < \gamma)$ yields a sequence $\{w_n\}$. We shall construct a function f_γ satisfying for each n the conditions

Received September 18, 1963.
This note was written with support from NSF Grant 688.