REMARKS ON A PAPER OF L. CESARI ON FUNCTIONAL ANALYSIS AND NONLINEAR DIFFERENTIAL EQUATIONS

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A well-known technique in dealing with oscillations of weakly nonlinear systems, that is, systems of the form

$$\dot{x} = Ax + \varepsilon f(t, x) \tag{1}$$

(ε a small parameter), is to set up certain transcendental equations (determining equations or bifurcation equations), the solutions of which are related to the periodic solutions of (1). One way of establishing such relations has been developed in recent years by Cesari, Hale and others; a comprehensive account of this method has just been published [4] (see also [2; Section 8.5]). The reader will find in the introduction of [4] references to literature and also a survey comparing the different ways which are known to define bifurcation equations.

If the matrix A in (1) is 0, the formal aspect of the Cesari-Hale method becomes quite transparent, so that the question arises whether a suitable generalization will allow elimination of the condition that ε be small. Cesari has in fact devised such a generalization and has obtained [1] a finite system of determining equations for arbitrary systems

$$\dot{x} = f(x, t),$$

where f is defined and satisfies certain smoothness-conditions in a finite region X of the (x, t)-space. The number of these determining equations depends upon such quantities as the bound for $|f|$ and Lipschitz constants.

In the case of a single differential equation, Cesari's method can roughly be described as follows (a more thorough report is given in Chapter 11 of [4]; another summary is given in Chapter 11 of [2]):

One associates with the differential equation $\dot{x} = f(x, t)$ a certain operator F, which depends on a finite number of parameters $a_1, \cdots, a_N$. The operator F is in some respect a modification of the operator $\xi \to \int^t f(\xi(t), t)dt$ used in existence and uniqueness proofs. The modification achieves two purposes: (1) F has the contraction property on a conveniently chosen function space over a prescribed interval [0, T] (whereas in the usual case the interval has to be chosen properly); (2) a periodic function $\xi(t)$ in the space (that is, a function with $\xi(0) = \xi(T)$) is mapped into a periodic function.

As a contraction operator F has a unique fixed element $\xi(a, t)$, which depends upon the parameters $a = (a_1, \cdots, a_N)$ and which can be constructed in the usual way by iteration. It turns out that $\xi(a, t)$ is periodic and satisfies a modified differential equation of the form

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