CONCERNING THE ORDER STRUCTURE OF KÖTHE SEQUENCE SPACES

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1. INTRODUCTION

Since 1934 a rather extensive study has been made, principally by G. Köthe, of the linear and topological structure of certain vector spaces λ of real sequences for which there is associated a dual sequence space λ^*. An excellent account of the basic theory of these spaces can be found in Section 30 of [4]. In 1951, this theory was extended to spaces of functions by J. Dieudonné [2]. These spaces are partially ordered in a natural way, and the purpose of this paper is to study some of the relations that exist between their order and topological structures. More specifically, we shall concern ourselves with some of the properties of the coarsest and finest compatible topologies (see Section 3) on these spaces. Since some of our results hold only for the sequence space case, we shall place our presentation in this framework and then indicate what modifications must be made for function spaces.

2. PRELIMINARY MATERIAL

Suppose ω denotes the ordered vector space of all sequences $x = (x_i)$ of real numbers and that ϕ denotes the linear subspace of ω consisting of those sequences with only finitely many non-zero components. (The linear operations and order relations on ω are defined in the usual coordinatewise fashion; for example, $x \geq y$ if $x_i \geq y_i$ for all i.) If λ is a linear subspace of ω containing ϕ, the *α-dual* λ^* of λ is defined by

$$\lambda^* = \{u = (u_i) \in \omega: \sum |x_i u_i| < +\infty \text{ for all } x = (x_i) \in \lambda\}.$$

(In this paper, Σ will indicate summation over the index set of all natural numbers.) The spaces λ and λ^* form a dual system $<\lambda, \lambda^*>$ with respect to the bilinear form $(x, u) \to <x, u> = \Sigma x_i u_i$. If $K = \{x \in \lambda: x_i \geq 0 \text{ for all } i\}$ denotes the cone in λ, the dual cone $K' = \{u \in \lambda^*: <x, u> \geq 0 \text{ for all } x \in K\}$ coincides with the cone of all sequences in λ^* with non-negative components.

The α-dual of λ^*, which we shall denote by λ^{**}, contains λ; if $\lambda = \lambda^{**}$, then λ is said to be *perfect*. For a given $s = (s_i)$ define

$$\lambda_s = \{u \in \omega: \sum |s_i u_i| < +\infty\};$$

then $\lambda = \bigcap_{v \in K'} \lambda_v$ if λ is perfect. Moreover,

$$\lambda = \bigcup_{y \in K} \lambda_y^*$$

if λ is solid (that is, if $|x| \leq |y|$ and $y \in \lambda$ imply $x \in \lambda$, where $|x| = (|x_i|)$ denotes the lattice theoretic absolute value of x in λ).

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