THE LATTICE-ORDERED GROUP OF AUTOMORPHISMS OF AN ORDERED SET

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1. INTRODUCTION

Let S be a totally ordered set, and let G be the group of all functions $f: S \to S$ such that f is one-to-one, onto, and the inequality $x < y$ $(x, y \in S)$ implies that $xf < yf$. We call such a function an *automorphism* of S. If f and g are automorphisms of S, then define $f \leq g$ if $xf \leq xg$ for all $x \in S$. It is well known and easily proved that this defines a partial order on G under which G is a lattice-ordered group (ℓ-group). For example, Problem 95 in [2] asks what ℓ-groups can be constructed in this way. In Section 2 we give a partial answer to this question by showing (Theorem 2) that any ℓ-group can be embedded in the ℓ-group of automorphisms of an appropriate ordered set. (*Ordered* means here, and throughout the paper, *totally ordered.*) The main embedding theorem (Theorem 1) gives more precise information on the embedding and suggests a more concrete formulation of Birkhoff's problem as follows: What ℓ-groups are *transitive* groups of automorphisms of ordered sets? The answer to this question is given by Theorem 3. Section 3 contains an application of the main embedding theorem. We prove that every ℓ-group can be embedded in a divisible ℓ-group. In Section 4, as an illustration of the techniques involved, we investigate the structure of the ℓ-group of permutations of the real line.

Notation. All groups will be written multiplicatively, and (most) functions will be written on the right. Thus if G is a group of permutations of a set S, if $f, g \in G$, and if $x \in S$, then fg is the function whose value at x is sometimes denoted by $g(f(x))$.

2. THE EMBEDDING THEOREM

Lemmas 1 through 4 are generalizations of lemmas that are well known if the subgroups under consideration are ℓ-ideals. In particular, the proofs of Lemmas 1 and 2 are sufficiently similar to the standard proofs (Birkhoff [2]) that they are omitted here.

Throughout the paper, G denotes an ℓ-group. A subgroup of G, which is also a sublattice, is an *ℓ-subgroup.* A subgroup C of G is *convex,* provided C contains along with any $x \geq 1$ also all y such that $x \geq y \geq 1$. For $x \in G$, $|x| = x \vee x^{-1}$. (The symbols $\vee$ and $\wedge$ denote the lattice operations.)

LEMMA 1. *Let* C *be a convex ℓ-subgroup of* G, *and let* $1 \leq a \in G$. *Define* $C^*(a) = \{x \in G \mid a \wedge |x| \in C\}$. *Then* $C^*(a)$ *is a convex ℓ-subgroup of* G *and* $C \subseteq C^*(a)$.

LEMMA 2. *Let* C *be a convex subgroup of* G. *Let* $R(C) = \{Cg \mid g \in G\}$ *be the set of all right cosets of* C *in* G. *If we define* $Cg \leq Ch$ *to mean there exists* $c \in C$

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