TRANSITION GRAPHS AND THE STAR-HEIGHT OF REGULAR EVENTS

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1. INTRODUCTION

Kleene [3] was the first to introduce the concept of regularity for expressions and sets of words. Others, including Copi, Elgot, and Wright [2], Myhill [6], and McNaughton and Yamada [5] have discussed this topic and its relation to finite automata. (We refer the reader especially to [2] for a presentation of regularity which is similar to ours.) We show that there exist regular sets of arbitrarily large star-height. Our second main result yields as a corollary an analysis theorem for finite automata which provides an upper bound for star-height of the behavior of the automaton in terms of the "cycle complexity" of the automaton's state graph. Our first result is then used to show that in some sense this latter result is best possible.

In Section 2 we define our concepts, introduce some notation, and prove some preliminary results. In Section 3, we show that for each positive integer n, there exists a regular set of star-height n. In fact, we obtain certain sufficient conditions for a regular set to have star-height greater than or equal to n. Some related results are also obtained. In Section 4, we show that for any directed graph of cycle rank n, the set of paths between any two points is a regular set of star-height no greater than n. We also show that equality is possible. In Section 5, we give a reasonable definition of feedback for a finite automaton, and we relate it to star-height.

2. PRELIMINARIES

In this section we give some definitions, introduce some notation and conventions, and prove some preliminary results.

Let $\mathscr{A}$ be a finite set of objects, say $\mathscr{A} = \{A_1, A_2, \cdots, A_n\}$. Let $\mathscr{A}^*$ be the free semigroup with identity generated by $\mathscr{A}$, where we write the operation multiplicatively and denote it by juxtaposition. We denote by A^m the element $\underbrace{AAA \cdots AA}_{m \text{ times}}$.
Let θ denote the identity element, so $\theta A_i = A_i \theta = A_i$ for all $i = 1, \cdots, n$. We shall also call $\mathscr{A}^*$ the set of words on the alphabet $\mathscr{A}$ and hence call an element of $\mathscr{A}^*$ a word (on the alphabet $\mathscr{A}$).

Let $\mathscr{S} = \mathscr{A} \cup \{\Lambda, \Theta\}$, where Λ and Θ are distinct objects not in $\mathscr{A}$. Let "$\vee$" and "$\cdot$" be associative binary operations and "$*$" a unary operation, and define $\mathfrak{S} = (\mathscr{S}, \vee, \cdot, *)$ to be the free algebra (cf. Birkhoff [1, p. vii]) generated by $\mathscr{S}$ with operations $\vee$, $\cdot$, and $*$. Thus $\mathscr{S} \subseteq \mathfrak{S}$, and $\sigma, \omega \in \mathfrak{S}$ implies $\sigma \vee \omega$, $\sigma \cdot \omega$, and

Received March 5, 1963.
This research was supported by funds supplied by the Office of Naval Research through Contract No. Nonr-1224(21) and the Office of the Chief Signal Officer through Contract No. DA-SIG-36-039-61-G4.