SOME CHARACTERIZATIONS OF THE LAGUERRE AND HERMITE POLYNOMIALS

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1. Recently Carlitz [1] proved the following representation for the Laguerre polynomials (for its definition see [2, p. 200]):

$$L_n^{(\alpha)}(x) = (n!)^{-1} \prod_{j=1}^{n} (xD - x + \alpha + j) \cdot 1 \tag{1.1}$$

where $D = d/dx$.

Let b_n $(n = 1, 2, 3, \cdots)$ be a sequence of numbers, and let

$$p_n(x) = \prod_{j=1}^{n} (xD + x + b_j) \cdot 1 \qquad (n = 1, 2, 3, \cdots), \tag{1.2}$$

$$p_0(x) = 1.$$

Obviously $p_n(x)$ is of degree exactly n. We propose here to prove the following theorem.

THEOREM 1. *If the set* $\{p_n(x)\}$, *defined by means of* (1.2), *is a set of orthogonal polynomials* [2, p. 147], *then* $p_n(x)$ *is the* n*th Laguerre polynomial.*

Proof. Since the set $\{p_n(x)\}$ is simple and orthogonal and since the coefficient of x^n in $p_n(x)$ is one, there is a three-term recurrence relation [2, p. 151]

$$\begin{aligned} p_{n+1}(x) &= (x + B_n)p_n(x) + C_n p_{n-1}(x) \qquad (n \geq 0), \\ p_{-1}(x) &= 0, \ p_0(x) = 1, \ C_n \neq 0. \end{aligned} \tag{1.3}$$

We see from (1.2) and (1.3) that

$$\begin{aligned} p_{n+1}(x) &= (xD + x + b_{n+1})p_n(x) = (x + n + b_{n+1})p_n(x) + (xD - n)p_n(x) \\ &= (x + B_n)p_n(x) + C_n p_{n-1}(x). \end{aligned} \tag{1.4}$$

Since $(xD - n)p_n(x)$ is a polynomial of degree $n - 1$, it follows that

$$B_n = n + b_{n+1} \tag{1.5}$$

and

$$(xD - n)p_n(x) = C_n p_{n-1}(x). \tag{1.6}$$

We also get from (1.2) and (1.3), respectively, the relations

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