A BASIS THEOREM FOR CUSP FORMS ON GROUPS OF GENUS ZERO

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1. Let $\Gamma(1)$ denote the modular group, that is, the group of linear fractional substitutions $w = (az + b)/(cz + d)$ with a, b, c and d integers and $ad - bc = 1$. Let μ denote the dimension of the vector space of cusp forms of dimension $-r$ for $\Gamma(1)$. (See Section 2 for definitions.) It is known that if r is an even integer, then $\mu = [r/12]$ if $r \not\equiv 2 \mod 12$ and $\mu = [r/12] - 1$ if $r \equiv 2 \mod 12$. In 1939 Petersson [6] proved the following theorem: *The first* μ *Poincaré series* (see (2.3)) $g_{-r}(z, 1)$, $g_{-r}(z, 2)$, $\cdots$, $g_{-r}(z, \mu)$ *span the vector space of cusp forms of dimension* $-r$ *for the modular group, where* $r > 2$ *is an even integer.* The object of this paper is to show that essentially the same proof applies to all real zonal horocyclic groups of genus zero and finite signature. The modular group is such a group.

To our knowledge this theorem is not implied by any of the more recent basis theorems. The proof fails if the genus of the group is larger than zero. In the next section we make the necessary definitions.

2. A *horocyclic* group Γ is a group of linear fractional transformations which maps a disk or half-plane one-to-one onto itself, is discontinuous at each interior point of the disk and is not discontinuous at any of the boundary points. (Equivalent terms are *Fuchsian group of the first kind* and *Grenzkreisgruppen* [9].) A *real* horocyclic group maps the upper half-plane onto itself; furthermore, if it possesses a translation it is termed *zonal.* The signature of a discontinuous group is an $n + 2$-tuple describing certain characteristics of a fundamental region; namely, the genus g, the number n of inequivalent fixed points (with respect to Γ) and an integer (possibly infinite) at least 2 associated with each of these fixed points. In case n is finite we write $(g, n; k_1, k_2, \cdots, k_n)$ for the signature. At an elliptic fixed point we associate the order of the transformation in Γ fixing the point; at a parabolic fixed point we associate the number ∞. If Γ is a zonal horocyclic group which has a finite signature $(g, n; k_1, k_2, \cdots, k_n)$, then we may assume that

$$2 \le k_1 \le k_2 \le \cdots \le k_s < \infty, \; k_{s+1} = \cdots = k_n = \infty, \quad \text{and } s < n.$$

(For a discussion of the signature of discontinuous groups and related matters see [1; pp. 203-209], [4] and [5; Ch. VII].) With few exceptions the converse is true: given

$$(g, n; k_1, k_2, \cdots, k_n), \; 2 \le k_1 \le k_2 \le \cdots \le k_s < \infty,$$

$$k_{s+1} = \cdots = k_n = \infty, \quad \text{and } s < n,$$

there exists a zonal horocyclic group Γ with the given signature. Since we are concerned with $g = 0$, we need only require that $n > 3$ or if $n = 3$ then $1/k_1 + 1/k_2 + 1/k_3 < 1$ [5].

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