ISOMETRIC IMMERSIONS OF CONSTANT CURVATURE MANIFOLDS

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INTRODUCTION

Let M^d and $\overline{M}^{d+k}$ be complete, differentiable (C^∞) Riemannian manifolds of constant sectional curvature C and $\overline{C}$ respectively. It is known that for $C < \overline{C}$ and $k < d - 1$, there exist no isometric immersions of M^d in $\overline{M}^{d+k}$. On the other hand, if $C \geq \overline{C}$ there are many such immersions, even for $k = 1$. We shall investigate the character of these immersions in the critical case $C = \overline{C}$, using a refinement of the method applied to the flat case in [2]. The general idea is that if M^d is not totally geodesic in $\overline{M}^{d+k}$, it must be bent along rather special submanifolds. We prove a precise formulation of this in the next section, and draw some consequences from it in Section 3.

THE MAIN THEOREM

Let M^d *and* $\overline{M}^{d+k}$ *be manifolds with the same constant curvature* C, M^d *being assumed complete. We assume further that* ψ: $M^d \to \overline{M}^{d+k}$ *is an isometric immersion, with* $k < d$. Our notation will be essentially that in [2]. In particular, we express the second fundamental form information of ψ in terms of a tensor T related to the classical operators S_z by the identity $\langle T_x(y), z\rangle = \langle S_z(x), y\rangle$, where $x, y \in M_m$ and $z \in (M_m)^\perp$. (Here $(M_m)^\perp$ denotes the orthogonal complement of $d\psi(M_m)$ in $\overline{M}_{\psi(m)}$.)

If $m \in M$, let $\mathcal{N}(m)$ be the space of null-vectors at m, that is, the subspace of M_m consisting of all vectors x such that $T_x = 0$. There is a useful result (Theorem 2 of [1]) which, though stated for the flat case, applies also in the case at hand. It asserts that *for each point* $m \in M$, *there exists a vector* $y \in \mathcal{N}(m)^\perp$ *such that* T_y *is one-one on* $\mathcal{N}(m)^\perp$. (Here the orthogonal complement is only in M_m.)

Let n be the minimum value of the dimension of $\mathcal{N}(m)$ on M, and let G be the (open) set of M on which this minimum occurs. Then $\mathcal{N}$ is a differentiable field of n-planes on G. Using this notation, we can state our main result.

THEOREM 1. *The field* $\mathcal{N}$ *is integrable on* G; *its leaves are complete, totally geodesic,* n*-dimensional submanifolds of* M^d, *with* $n \geq d - k$. *Furthermore, each leaf is totally geodesic in* $\overline{M}^{d+k}$ *relative to* ψ.

Proof. The last assertion follows immediately from the definition of $\mathcal{N}$. The lower bound for n is a consequence of the theorem of Chern and Kuiper stated above. The proof that $\mathcal{N}$ is integrable on G and that its leaves are totally geodesic is the same as in the flat case. This is true because the proof involves only the relative position of $\psi(M)$ in $\overline{M}$, that is, involves only the second fundamental form and Codazzi equation of ψ. However, the essential feature of the theorem is the *completeness* of the leaves. This depends not merely on relative information, but also

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