UNIFORM DISTRIBUTION RELATIVE TO A FIXED SEQUENCE

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1. INTRODUCTION

In the usual theory of the distribution modulo 1 of an increasing sequence $s_1, s_2, \cdots$ of real numbers, one considers the positions of the successive terms of the sequence in the unit intervals $(n, n+1)$ into which they fall. These positions are specified by the fractional parts of $s_1, s_2, \cdots$. The definition of uniform distribution modulo 1, in terms of these fractional parts, is well known. In an earlier paper [1], one of us considered a generalization of this concept, in which the unit intervals are replaced by the intervals (z_n, z_{n+1}) between the successive numbers of a fixed sequence $0 < z_1 < z_2 < \cdots$, where $z_n \to \infty$ with n. The fractional part of a positive real number t, relative to the sequence $\Delta = \{z_n\}$, is defined by

(1) $$<t>_\Delta = \frac{t - z_{n-1}}{z_n - z_{n-1}} \quad \text{for } z_{n-1} \leq t < z_n .$$

A sequence $s_1, s_2, \cdots$ is said to be *uniformly distributed modulo* Δ if the proportion of $s_1, \cdots, s_N$ for which $<s_k>_\Delta < \alpha$ has the limit α as $N \to \infty$, for each α such that $0 < \alpha < 1$.

It is reasonable to impose some condition on the sequence Δ, and we shall suppose that $z_n - z_{n-1}$ is either monotonic increasing or monotonic decreasing. In the increasing case, it was proved in [1] that the sequence $s_k = kx$ is uniformly distributed modulo Δ for each $x > 0$ provided that $z_n/z_{n-1} \to 1$ as $n \to \infty$, and that this supplementary condition is necessary. The decreasing case is more difficult; it was proved that the sequence $s_k = kx$ is uniformly distributed modulo Δ for almost all $x > 0$ (in the sense of Lebesgue measure) provided that $z_n - z_{n-1} = O(z_n^{-1})$. But this was a severe restriction on the z_n.

The main object of the present note is to prove this "almost all" result in the decreasing case without imposing any additional condition on the z_n. Although the case $s_k = kx$ is the one we have principally in mind, the method yields a more general result with little extra effort. We prove the following result. [The words "increasing" and "decreasing" are used in the wide sense henceforth.]

THEOREM. *Suppose that* $z_n - z_{n-1}$ *decreases as* n *increases, and that* $z_n \to \infty$. *Let* $a_1, a_2, \cdots$ *be any sequence of positive real numbers such that*

(2) $$a_{k+1} - a_k \geq Ca_k/k \quad (C > 0).$$

Then the sequence $s_k = a_k x$ *is uniformly distributed modulo* $\Delta = \{z_n\}$ *for almost all* $x > 0$. *In particular, this holds for* $s_k = kx$ *or, more generally, for* $s_k = k^\gamma x$ *for any fixed* $\gamma > 0$.

We may remark that the condition (2) is also satisfied if $a_{k+1} - a_k$ increases with k.

The proof of the theorem makes use of the condition, given in the preceding note, for a sequence $s_k(x)$ to be uniformly distributed (mod 1) for almost all x in an interval (α, β).

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