A REMARK ON RINGS AND ALGEBRAS

I. N. Herstein

We have recently been concerned with conditions which, when imposed on a ring, render the commutator ideal of this ring to be a nil ideal [2, 3]. A standard tool for exhibiting nil ideals is the following result of Amitsur [1]: If A is a finitely-generated *algebra* over a field, and A satisfies a polynomial identity, then the (Jacobson) radical of A is a nil ideal.

This above-cited theorem of Amitsur has one shortcoming in that it has not as yet been established for rings, while the natural area of its application is to rings. For this reason we prove the metamathematical theorem, given below, which allows us to transfer a certain class of results from algebras (in which Amitsur's theorem would play a significant role) to arbitrary rings.

As applications of the principal result of this note, we shall give a simplified proof of a theorem which we have recently proved by more complicated means [3] and we complete the proof, for rings, of the result proved in [2]. In [2] we applied Amitsur's theorem too widely, namely to rings; the proof as given in [2], however, is valid only for algebras. By the theorem to be proved, it automatically then becomes valid for rings as well.

A ring R is said to be of characteristic 0 if whenever $mx = 0$ with $x \neq 0$ in R and m an integer then $m = 0$. Let R be of characteristic 0, and suppose that $M = \{(x, n) \mid x \in R, n \neq 0 \text{ any integer}\}$; in M equality is defined, as usual, componentwise. Given (x_1, n_1), (x_2, n_2) in M we define $(x_1, n_1) \sim (x_2, n_2)$ if $n_2 x_1 = n_1 x_2$. It is immediate that this defines an equivalence relation on M. Let R* be the set of equivalence classes of M; if $[x, n]$ denotes the equivalence class of (x, n), then, since R is of characteristic 0, it follows easily that addition and multiplication defined by $[x_1, n_1] + [x_2, n_2] = [n_2 x_1 + n_1 x_2, n_1 n_2]$ and $[x_1, n_1][x_2, n_2] = [x_1 x_2, n_1 n_2]$ are well defined operations in R* under which R* becomes a ring containing an isomorphic copy of R. Moreover, R* is an algebra over the rational field. We call R* the rationalization of R.

One final bit of notation: for any ring R let C(R) denote the commutator ideal of R. We proceed to our theorem.

THEOREM. *Let* P *be a property defined on rings such that:*

(1) *if* P(R) *is true, then so are* P(U) *and* P(R/U) *for any (two-sided) ideal* U *of* R.

(2) *if* R *is of characteristic* 0 *and if* P(R) *is true, then so is* P(R*).

(3) *if* A *is an algebra over a field for which* P(A) *is true, then* C(A) *is a nil ideal.*

Then if P(R) *is true for any ring* R, C(R) *must be a nil ideal.*

Proof. Let R be a ring in which P(R) is true. We claim that, without loss of generality, we may assume that R has no non-zero nil ideals. For, if N is the

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