ON THE MODULUS OF SMOOTHNESS AND DIVERGENT SERIES IN BANACH SPACES

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1. INTRODUCTION

The notions of uniform smoothness and of the modulus of smoothness were introduced by Day [4]. He proved that a Banach space X is uniformly smooth if and only if X^* is uniformly convex and gave estimations of the modulus of smoothness of X in terms of the modulus of convexity of X^*. Later other, but equivalent, moduli of smoothness were introduced; see, for example, Köthe [13, pp. 366-367]. In Section 2 we evaluate the exact value of such a modulus of smoothness of a Banach space in terms of the modulus of convexity of its conjugate, and we apply this result to show that inner product spaces are characterized by being the "smoothest" spaces. In Section 3 we prove a result on divergent series in uniformly smooth Banach spaces, which is in a certain sense dual to a result of Kadec [11].

In Section 4 we apply the result of Section 3 to prove that if the moduli of convexity and smoothness of X behave asymptotically like those of ℓ_2 and if X has an unconditional basis, then X is isomorphic but, in general, not isometric to ℓ_2. Geometrically, this result has the following, rather surprising, formulation (ignoring for the moment the requirement concerning the basis): If the unit cell S of a Banach space is sufficiently smooth and convex (smoothness and convexity are measured by the respective moduli), then there is in the space a convex body S_1, whose smoothness and convexity are the greatest possible, such that $S \subset S_1 \subset kS$ for some $k \geq 1$. However, this S_1 cannot in general be chosen to be very close to S (for example, the best possible k may be arbitrarily large for suitable S).

In the last section we introduce two indices which classify Banach spaces in terms of convergent or divergent series in them.

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Notations. We consider only Banach spaces of dimension at least 2. Let X be a Banach space. Its modulus of convexity is defined by

$$\delta_X(\varepsilon) = \frac{1}{2} \inf_{\substack{\|x\|=\|y\|=1 \\ \|x-y\|=\varepsilon}} (2 - \|x+y\|) \qquad (0 \leq \varepsilon \leq 2).$$

X is called uniformly convex if $\delta_X(\varepsilon) \geq 0$ for every $\varepsilon > 0$. The modulus of smoothness of a space X is defined by

$$\rho_X(\tau) = \frac{1}{2} \sup_{\|x\|=1,\ \|y\|=\tau} (\|x+y\| + \|x-y\| - 2) \qquad (\tau \geq 0).$$

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