THE ALGEBRA OF SEMIPERIODIC SEQUENCES

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A sequence $z = \{z_n\}$ of complex numbers is called periodic if there exists positive m such that $z_{m+n} = z_n$ for all n. A sequence is called semiperiodic if it is in the uniform closure of the space of periodic sequences.

In a recent article [2], A. Wilansky and the author discussed the Banach space Q of semiperiodic sequences. There we mentioned that Q is not the space of almost-periodic functions on the integers.

In the present article, we put the obvious Banach algebra structure on Q and show that $Q = C(\bar{\omega})$, where $\bar{\omega}$ is the character group of R^0, where R^0 denotes the additive group of rationals mod 1 in the discrete topology. Hence, the theory of almost periodic functions on topological groups becomes available to us. The general problem of Bohr compactifications of locally compact abelian groups has been discussed by H. Anzai and S. Kakutani in [1], in which paper it is proved that $\bar{\omega}$ (there called the *universal monothetic Cantor group*) can be obtained as a Bohr compactification of the group of all integers.

In [2], A. Wilansky and the author showed that any matrix summing Q was bounded in the usual matrix norm. Here, we give a characterization of the matrices summing Q in terms of sequences of measures on $\bar{\omega}$.

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1. THE TOPOLOGICAL GROUP $\bar{\omega}$

Let (τ, ρ) denote the additive semi-group of positive integers τ with the metric ρ, where ρ is defined as follows:

$$\rho(x, y) = \frac{1}{n} \text{ if } n! \text{ divides } |x - y| \text{ and } (n+1)! \text{ does not, } \rho(x, x) = 0.$$

Let $\bar{\omega}$ denote the completion of this metric space. It is easily verified that $\bar{\omega}$ is a compact topological group with metric ρ and addition + inherited from τ, for example, the identity 0 is the limit of the Cauchy sequence $\{n!\}$.

THEOREM 1. *Let* R^0 *denote the additive group of rationals* mod 1 *in the discrete topology. Then* $\bar{\omega}$ *is the character group of* R^0.

Proof. Let $x = \{x_n\} \in \bar{\omega}$. Then for $r \in R^0$, we define a homomorphism of R^0 into the circle group by

$$x(r) = \lim_{n \to \infty} \exp 2\pi i x_n r .$$

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