SUMS OF NORMAL FUNCTIONS AND FATOU POINTS

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Let C and D denote the unit circle and open disk, respectively. If f(z) is a complex valued function defined in D, the outer angular cluster set f(z) at a point $e^{i\theta}$ in C is denoted by $C_A(f, e^{i\theta})$ [6, p. 69], while the radial cluster set of f(z) at $e^{i\theta}$ is denoted by $C_R(f, e^{i\theta})$. The non-Euclidean hyperbolic distance between points z and z' in D is denoted by $\rho(z, z')$ [3, Chapter 2].

Bagemihl and Seidel have shown that every normal holomorphic function in D has a Fatou point [1, Theorem 4], and, in fact, that the set of Fatou points is dense on C [2, Corollary 1]. However, the author has shown that the sum of two normal holomorphic functions need not be normal [5, Theorem 4]. It is our present purpose to show that the sum of two normal holomorphic functions need not have a Fatou point.

We first prove a lemma concerning a Blaschke product.

LEMMA. *Let* E *be a prescribed countable set in* C. *Then there exists a Blaschke product* B(z) *such that*

(1) *for every* $e^{i\theta} \in E$, B(z) *has infinitely many zeros on the radius to* $e^{i\theta}$; *and*

(2) *there exist sequences* $\{R_n\}$ *and* $\{S_n\}$ *of real numbers, with* $0 < R_n < S_n < R_{n+1} < 1$, *such that* $|B(w_n)| \to 1$ *for every sequence* $\{w_n\}$ *with* $R_n < |w_n| < S_n$.

Proof. Let $a_n = 1 - 2^{-n}$ $(n = 1, 2, \cdots)$; and let $\{e^{i\theta_n}\}$ be an enumeration of the elements of E, with every element of E appearing infinitely often in the enumeration.

We shall now locate the zeros $\{z_n\}$ of the Blaschke product. Set $z_1 = \frac{1}{2}e^{i\theta_1}$. Let R_1 be chosen with $|z_1| < R_1 < 1$ such that

$$\left|\frac{z - z_1}{1 - \bar{z}_1 z}\right| > a_1 \qquad (|z| > R_1).$$

Now choose S_1 with $R_1 < S_1 < 1$, and then choose $z_2 \in D$ such that $|z_2| > S_1$, $\arg z_2 = \theta_2$, and

$$\frac{z_2 - z}{1 - \bar{z}_2 z} > a_2 \qquad (|z| < S_1).$$

We now proceed inductively. Assume $z_1, z_2, \cdots, z_n$; $R_1, R_2, \cdots, R_{n-1}$; and $S_1, S_2, \cdots, S_{n-1}$ have been chosen such that

(3) $$\arg z_j = \theta_j \qquad (1 \leq j \leq n),$$

(4) $$|z_j| < R_j < S_j < |z_{j+1}| \qquad (1 \leq j \leq n - 1),$$

(5) $$\left|\frac{z_{j+1} - z}{1 - \overline{z_{j+1}} z}\right| > a_{j+1} \qquad (|z| < S_j;\ 1 \leq j \leq n - 1),$$

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