SUFFICIENT CONDITIONS FOR SEMICONTINUOUS SURFACE INTEGRALS

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1. INTRODUCTION

Semicontinuous parametric and nonparametric line integrals were introduced by L. Tonelli [14] to aid in establishing existence theorems in the calculus of variations. Following this procedure, McShane [9, 10] proved the first theorems on semicontinuous parametric surface integrals for surfaces having Lipschitzian representations. Subsequently, T. Rado [12] and L. Cesari [3] proved successively stronger theorems (slightly later, but independently, P. V. Reichelderfer [13] published an intermediate result; see also G. M. Ewing [7]). Cesari's theorems deal with arbitrary surfaces of finite area defined on the unit square.

A surface (T, Q) defined on the unit square Q is a mapping T of

$$Q = \{(u, v): 0 \leq u,\ v \leq 1\}$$

into Euclidean three-space $E^3 = \{(x, y, z)\}$. All mappings will be supposed continuous in this paper. We shall say that (T, Q) is of bounded variation or has BV if it has finite area.

We shall call a function of six variables $f(x, y, z, J_1, J_2, J_3) = f(p, J)$ a parametric integrand if it is continuous and positively homogeneous in J. If a BV surface (T, Q) is absolutely continuous, then generalized Jacobians (see [5]) exist almost everywhere in Q, and if $f(p, J)$ is a parametric integrand bounded on $T(Q)$, the Lebesgue-Tonelli integral

$$I(T, Q) = (Q) \int f(T(w), J(w))\, du\, dv \tag{1}$$

exists, where $J(w) = (J_1, J_2, J_3)$ are the generalized Jacobians. This is the integral used by Cesari. The other authors also used this integral; but they restricted themselves to mappings (T, Q) that have sufficiently well behaved Jacobians in the usual sense.

Bouligand [1] showed that in the study of semicontinuous parametric line integrals $\int f(x, \dot{x})dt$, the usual conditions on the Weierstrass function can be replaced by weaker conditions involving the convexity of $f(x, \dot{x})$ in $\dot{x}$ for every x. Later Aronszain (as reported by Pauc [11]) did the same for nonparametric line integrals.

Here, we shall continue the work of McShane, Rado, Reichelderfer, and Cesari on surface integrals, generalizing their results in the following respects:

(1) The domain A of the surface (T, A) will be an arbitrary admissible set, as defined by Cesari [5] and given below;

(2) Absolutely continuous representations of the surfaces will not be used. We shall use the integral defined by Cesari [2], in the extended form given by Cesari and Turner [6], for which no special representation is necessary;

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