ON A THEOREM OF FISHER CONCERNING THE HOMEOMORPHISM GROUP OF A MANIFOLD

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An n-manifold M^n is a connected, separable metric space each point of which has an open neighborhood whose closure is homeomorphic to the n-cell I^n. An *internal cell* of M^n is a subset Q of M^n for which there exists a homeomorphism of Euclidean space E^n into M^n such that Q is the image of the unit n-cell of E^n. Alternatively, Q is a topological n-cell in the interior $\mathring{M}^n$ of M^n whose boundary $\dot{Q}$ is locally flat in M^n [1]. A homeomorphism h of M^n is *supported* on a set $K \subset M^n$ if $h(x) = x$ whenever $x \notin K$. Suppose that $H(M^n)$ denotes the group of all homeomorphisms of M^n onto M^n and $FH(M^n)$ denotes the subgroup generated by homeomorphisms supported on internal cells. Then according to Fisher [2] $FH(M^n)$ *is simple and is the intersection of all nontrivial normal subgroups of* $H(M^n)$.

Suppose $\varepsilon > 0$ and $FH_\varepsilon(M^n)$ denotes the subgroup of $FH(M^n)$ generated by homeomorphisms supported on internal cells of diameter less than ε. The purpose of this note is to prove that

$$FH(M^n) = \bigcap_{\varepsilon > 0} FH_\varepsilon(M^n),$$

that is, a homeomorphism h is in $FH(M^n)$ if and only if for each $\varepsilon > 0$, h is the composition of homeomorphisms supported on subsets of the interior of M^n of diameter less than ε. A similar theorem holds for the piecewise linear case.

The following lemma has a straightforward proof.

LEMMA 1. *Let* $I^n = I^{n-1} \times I^1$ *and suppose* X *is a compact subset of* I^n *such that* $X \cap \dot{I}^n \subset I^{n-1} \times 0$. *Then there is a piecewise linear homeomorphism* h *of* I^n *such that* $h | \dot{I}^n = 1$ *and* $h(X) \subset I^{n-1} \times [0, 1/2)$.

LEMMA 2. *Let* h *be a homeomorphism of* $I^n = I^{n-1} \times I^1$ *onto itself such that* $h | \dot{I}^n = 1$ *and* $h(I^{n-1} \times 1/2) \subset I^{n-1} \times [1/3, 2/3]$. *Then there exists a homeomorphism* h' *of* I^n *such that*

$$h' | (\dot{I}^n \cup I^{n-1} \times [0, 1/4] \cup I^{n-1} \times [3/4, 1]) = 1 \quad \textit{and} \quad h' | I^{n-1} \times 1/2 = h | I^{n-1} \times 1/2 .$$

Proof. Let g be a piecewise linear homeomorphism of $I^{n-1} \times [1/4, 3/4]$ onto $I^{n-1} \times [0, 1]$ that is the identity on $I^{n-1} \times [1/2, 2/3]$. Let $h': I^n \to I^n$ be defined by

$$h'(x) = \begin{cases} x, & x \in I^{n-1} \times ([0, 1/4] \cup [3/4, 1]) \\ g^{-1}hg(x), & x \in I^{n-1} \times [1/4, 3/4] \end{cases}$$

Remark. If h is piecewise linear, so is h'.

LEMMA 3. *Let* $h: I^n \to I^n$ *be a homeomorphism such that* $h | \dot{I}^n = 1$. *Then* h *is the composition of five homeomorphisms, each the identity on* $\dot{I}^n$, *and each supported on one of the cells*

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