THE INTERSECTION OF FIBONACCI SEQUENCES

S. K. Stein

For the construction of a groupoid satisfying the identity $a((a \cdot ba)a) = b$ but not the identity $(a(ab \cdot a))a = b$, it was necessary to examine the intersection of Fibonacci sequences. In particular, Theorem 1 below was used. We shall prove that two Fibonacci sequences generally do not meet and that if they do meet at least three times, then one is simply the tail of the other.

If a_1 and a_2 are positive integers $(a_1 \leq a_2)$, let $F(a_1, a_2)$, or simply F, denote the Fibonacci sequence $a_1, a_2, a_3, \cdots$ whose first two terms are a_1 and a_2 (that is, $a_k = a_{k-1} + a_{k-2}$ for $k \geq 3$). Let $\overline{F}(a_1, a_2)$, or simply $\overline{F}$, be the set $\{a_1, a_2, a_3, \cdots\}$. We denote the kth term of the standard Fibonacci sequence $F(1, 1)$ by f_k. For convenience we define f_0 to be 0. A predicate P about positive integers holds "for almost all n" if $\{n: P(n) \text{ is true}\}$ has density 1, that is, if $\lim_{n\to\infty} A(n)/n = 1$, where $A(n)$ is the number of integers not exceeding n for which P is true.

THEOREM 1. *If* n *is a positive integer and* $F_1, F_2, \cdots, F_s$ *are Fibonacci sequences, then there is an integer* $m > n$ *such that* $\overline{F}(n, m) \cap \overline{F}_i$ *consists of at most the element* n, *for each* $i = 1, 2, \cdots s$.

Theorem 1 follows from the following stronger result.

THEOREM 2. *If* v_1 *is a positive integer and* F *is a Fibonacci sequence, then for almost all* v_2, $\overline{F}(v_1, v_2) \cap \overline{F}$ *consists of at most the element* v_1.

Proof. Let the first two terms of F be u_1, u_2. Since

$$\lim_{k\to\infty} u_{k+1}/u_k = (1 + \sqrt{5})/2,$$

there is an integer m such that $u_{k+1}/u_k < 2$ for all $k \geq m$. Let n_0 be one such m for which, in addition, $u_{n_0+1} - u_{n_0} > v_1$ and $u_{n_0} > v_1$. We write

$$u_{n_0} = M_1,\ u_{n_0+1} = M_2,\ \cdots,\ u_{n_0+j} = M_{j+1}.$$

Thus $F(M_1, M_2) = F(u_{n_0}, u_{n_0+1})$.

We shall determine an upper bound L for the number of v_2's $(M_1 \leq v_2 < M_2)$ such that $\overline{F}(v_1, v_2)$ meets $\overline{F}(M_1, M_2)$. This L will turn out to be small in comparison with $M_1 - M_2$, the number of v_2's for which $M_1 \leq v_2 < M_2$. From this, Theorem 2 follows easily.

First let us examine where $\overline{F}(v_1, v_2)$ might meet $\overline{F}(M_1, M_2)$. Since $v_1 < M_1$ and $v_2 < M_2$, induction implies that $v_k < M_k$ for all $k \geq 1$. Thus if $v_k \in \overline{F}(M_1, M_2)$ and $v_k = M_i$, then i must be less than k. On the other hand, we shall show that $v_k > M_{k-2}$ for $k \geq 3$. Indeed, $v_3 > M_1$ and

$$v_4 = v_2 + v_3 > M_1 + M_1 > M_2.$$

Received May 15, 1962.
This research was supported by the Air Force Office of Scientific Research.