NIL IDEALS IN GROUP RINGS

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Let R be a commutative ring having no nonzero nilpotent elements. We consider nil ideals in the group ring R[G]. In this paper conditions on G and R are found that ensure that Nil R[G], the upper nil radical of the group ring, is trivial. We also obtain necessary and sufficient conditions for the existence of nontrivial nilpotent ideals. In the particular case where R is a field K, the above results yield sufficient conditions on G and K for the semi-simplicity of the group algebra K[G]. These are similar to and, in fact, motivated by the results of S. A. Amitsur in [1] for fields of characteristic zero.

If there exists a positive integer $\overline{m}$ with $\overline{m}R = (0)$, then we set ch R, the characteristic of R, equal to the smallest such positive integer. Otherwise, ch R = 0. In the first case, let $\pi(R) = \{p_1, p_2, \cdots, p_k\}$ be the set of prime divisors of m = ch R. Then since R has no nontrivial nilpotent elements, it is immediate that each prime occurs only to the first power. The integers $m/p_1, m/p_2, \cdots, m/p_k$ are relatively prime, so there is a linear sum

$$n_1 m/p_1 + n_2 m/p_2 + \cdots + n_k m/p_k = 1\,.$$

This induces a decomposition

$$R = R_{p_1} \dotplus R_{p_2} \dotplus \cdots \dotplus R_{p_k}$$

of R as an internal direct sum of nonzero ideals, where

$$R_{p_i} = n_i m/p_i R = \{r \in R\colon p_i r = 0\}\,.$$

Hence, for any group G,

$$R[G] = R_{p_1}[G] \dotplus R_{p_2}[G] \dotplus \cdots \dotplus R_{p_k}[G].$$

An ideal of the group ring is then nil or nilpotent if and only if its projection into each factor is. This effectively reduces all ch R ≠ 0 considerations to the prime case.

We say an element $\sigma \in G$ is a p element if it is of order p^j for some $j \geq 1$.

THEOREM I. *Let* R *be a commutative ring having no nonzero nilpotent elements. Suppose that* ch R ≠ 0 *and that* G *has no* p *elements for all* $p \in \pi(R)$. *Then* Nil R[G] = (0).

First, we need a few lemmas. Let Γ be any ring. We write Comm Γ for the commutator of Γ, the set of all finite sums of elements of the form ab - ba with a, b ∈ Γ.

LEMMA 1. *Let* p *be a prime, and let* k *and* n *be arbitrary positive integers. Then for all* $x_1, x_2, \cdots, x_n \in \Gamma$,

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