SOME FREE PRODUCTS OF CYCLIC GROUPS

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Let p and q be positive integers. The principal question we wish to consider is that of giving a representation in terms of linear fractional transformations for the free product of a cyclic group of order p and a cyclic group of order q. These include the Hecke groups discussed in [1], which correspond to the choice $p = 2$, $q \geq 3$. In addition we consider two subgroups of the modular group and show that these are free products of cyclic groups. The method of proof we employ is elementary; it is patterned after the proof given by K. A. Hirsch in his appendix to the second volume of Kuroš's book on group theory [2], that Γ is the free product of a cyclic group of order 2 and a cyclic group of order 3. The referee points out that essentially the same proof is given by H. Rademacher in his paper [5].

We introduce some notation. For a positive integer n, define

$$\lambda_n = 2 \cos(\pi/n), \quad A_n = \begin{bmatrix} 0 & 1 \\ -1 & \lambda_n \end{bmatrix}, \quad B_n = \begin{bmatrix} 0 & -1 \\ 1 & \lambda_n \end{bmatrix}.$$

Then the eigenvalues of A_n and B_n are the numbers α_n, β_n, where

$$\alpha_n = \exp(i\pi/n), \quad \beta_n = \exp(-i\pi/n)$$

are primitive $(2n)^{th}$ roots of unity. If $n > 1$, these are distinct and both A_n and B_n are similar to diag (α_n, β_n). Thus the least positive integer k such that $A_n^k = \pm I$, $B_n^k = \pm I$ is $k = n$; and

$$A_n^n = B_n^n = -I.$$

We set $A = A_p$, $B = B_q$,

$$\Delta = \Delta_{p,q} = \{A, B\}$$

($\{A, B\}$ denotes the group generated by A and B) and agree to identify a matrix with its negative. This is equivalent to considering Δ as a group of linear fractional transformations (which, by the way, is a discontinuous group). Then to show that Δ is the free product of a cyclic group of order p and a cyclic group of order q it is only necessary to show that the relations

$$A^p = B^q = 1$$

are the defining relations for Δ. We assume that $p \geq 2$, $q \geq 3$. (The case $p = q = 2$ will be treated separately.) We set

(1) $$a_r = \frac{\alpha_p^r - \beta_p^r}{\alpha_p - \beta_p} = \frac{\sin(r\pi/p)}{\sin(\pi/p)},$$

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