PERIODIC, ALMOST-PERIODIC, AND SEMIPERIODIC SEQUENCES

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Dedicated by the second author to his teacher, C. W. Walmsley, who died March 27, 1962, in Derbyshire, England.

1. INTRODUCTION

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This article was suggested by two distinct lines of investigation, which merged after each had proceeded independently. These were the study of almost-periodic functions and sequences, and the study of matrix transformations of periodic sequences initiated by Vermes [6], [7] and Newton [5]. In the latter investigation, the the natural desire of functional analysts to work in a complete space (see, for example, Corollary 2) prompted us to deal with the completion of the space of periodic sequences. This completion turns out to be smaller than the space of almost-periodic sequences, thus damping any hopes of an extensive use of the theories of Bohr and von Neumann. The situation is this; each almost-periodic function on $(-\infty, \infty)$ is uniformly approximable by linear combinations of periodic functions. In attempting to restrict this result to the integers, we run into several snags. First, the restriction of a periodic function need not be a periodic sequence; second, the set of periodic sequences is already a linear space, so that its linear closure is the same as its closure; and, finally, that closure is actually smaller than the space of restrictions of almost-periodic functions, and therefore the analogous theorem is actually false.

A sequence x of complex numbers is called *semiperiodic* if to each $\varepsilon > 0$, there corresponds a positive integer n such that $|x_k - x_{k+rn}| < \varepsilon$ for all r, k; it is called *almost-periodic* if to each $\varepsilon > 0$, there corresponds a positive integer n such that every interval $(K, K + n)$, $K = 1, 2, \cdots$, contains an integer t satisfying the condition $|x_r - x_{r+t}| < \varepsilon$ for all r.

Taking m to be the familiar Banach space of all bounded complex sequences with $\|x\| = \sup |x_n|$, the closure in m of the set p of periodic sequences is shown, in Theorem 1, to be precisely the set q of semiperiodic sequences.

A related problem of interest concerns the properties of the space q as a subspace of m, considered as the conjugate of the space ℓ of absolutely convergent series with $\|x\| = \Sigma |x_n|$. The space q turns out to be large enough to be norming over ℓ. (See Section 5, below, for a definition of this concept.)

Since q is so large, it is interesting that there exist regular matrices which sum all of its sequences (for example, the Cesàro matrix, which yields the von Neumann mean) especially since known theorems place restrictions on the number of bounded divergent sequences summed by a regular matrix.

If S denotes the set of (C, 1)-summable bounded sequences, then $S \setminus c_0$ is norming over ℓ: this suggests the problem of identifying the matrices A having the

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