AN ANALYTICAL APPROACH TO THE DIFFERENTIAL EQUATIONS OF THE BIRTH-AND-DEATH PROCESS

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1. INTRODUCTION

This paper presents a purely analytical approach to the problems of uniqueness and existence of solutions (separate or simultaneous) for the forward and the backward differential equations of the so-called birth-and-death process. The conditions imposed on the solutions $\{p_{mn}(t)\}$ are of the types

$$p_{mn}(0) = \delta_{mn} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases} \tag{1.1}$$

$$p_{mn}(t) \geq 0, \tag{1.2}$$

$$\sum_n p_{mn}(t) \leq 1; \tag{1.3}$$

no attention is paid to the so-called semigroup property (compare [6] and [9]). The variable t is usually restricted to a fixed finite interval; this is definitely more general than the case $0 \leq t < \infty$ which is considered by most authors, mainly in order to permit use of the Laplace transform of $p_{mn}(t)$.

In our method of proof, neither uniqueness nor existence offers any great difficulty. The major effort is spent in obtaining, for many cases of interest, an explicit representation of all the possible solutions. In particular (see Theorem 9.3) such a representation is obtained for the case where the required solution is to satisfy both the forward and the backward differential equations, in addition to the conditions (1.1) to (1.3) above. An explicit representation for a certain subclass of such solutions in $0 \leq t < \infty$ was already obtained in the interesting paper [7] by Karlin and McGregor.

The conditions involved are highly redundant. In view of this, the problems on hand can be approached from many different directions. Our approach is probably closest to that of Arley and Borchsenius [1] and that of Reuter and Ledermann [10]. An approach involving the analytical theory of continued fractions was announced by Koopman [8]. A great variety of other methods, often for more general situations, can be found in the papers of Feller [2] to [6].

2. STATEMENT OF THE PROBLEM

In this paper, λ_n and μ_n ($n = 0, 1, 2, \cdots$) denote given nonnegative real numbers. By T we denote a fixed positive number (allowing occasionally that $T = +\infty$).

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