DIFFERENTIAL SYSTEMS AND EXTENSION OF LYAPUNOV'S METHOD

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Let I denote the half-line $0 \leq t < \infty$, and let R^n denote n-dimensional Euclidian space. We consider the differential systems

$$\left.\begin{array}{ll} (1) & x' = f(t, x); \;\; x(t_0) = x_0, \\ (2) & y' = g(t, y); \;\; y(t_0) = y_0, \end{array}\right\} \quad (t_0 \geq 0)$$

where x, y, f and g are n-dimensional vectors, and where the functions f(t, x), g(t, y) are defined and continuous on the product space $I \times R^n$. In Theorems 1 to 11 below, we establish a number of results on the stability and boundedness of solutions of the systems (1) and (2). Our results constitute an extension of work of Yoshizawa [7], [8], Brauer [1], and Conti [2].

We adopt the notation $R^+ = [0, \infty)$ and $|x| = \Sigma_{i=1}^n |x_i|$, and we shall write d(x, y) for $|x - y|$. Let a function $V(t, x, y) \geq 0$ be defined and continuous on the product space $I \times R^n \times R^n$, and suppose that it satisfies Lipschitz's condition in x and y locally. In particular, we assume that $V(t, x, x) \geq 0$ for (t, x) in $I \times R^n$. Following Yoshizawa [7], we next define the function

$$(3) \qquad V^*(t, x, y) = \limsup_{h \to 0^+} \frac{1}{h}[V(t + h, x + hf(t, x), y + hg(t, y)) - V(t, x, y)].$$

With respect to these functions we state the following lemmas.

LEMMA 1. *Let the function* W(t, r) *be defined and continuous on* $I \times R^+$. *Suppose further that the function* $V^*(t, x, y)$ *of* (3) *satisfies the condition*

$$(4) \qquad V^*(t, x, y) \leq W(t, V(t, x, y)).$$

Let r(t) *be the maximum solution of the differential equation*

$$(5) \qquad r' = W(t, r), \qquad r(t_0) = r_0 \geq 0.$$

If x(t) *and* y(t) *are any two solutions of* (1) *and* (2) *such that* $V(t_0, x_0, y_0) \leq r_0$, *then*

$$(6) \qquad V(t, x(t), y(t)) \leq r(t) \qquad (t \geq t_0).$$

LEMMA 2. *If the assumptions of Lemma* 1 *hold, except that the condition* (4) *is replaced by the inequality*

$$(7) \qquad A(t)\,V^*(t, x, y) + A^*(t)\,V(t, x, y) \leq W(t, A(t)\,V(t, x, y)),$$

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