AN EXISTENCE THEOREM FOR PERIODIC SOLUTIONS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

We consider a system of ordinary differential equations

$$\dot{x} = f(x, t) \qquad (x = (x_1, \cdots, x_n),\ f = (f_1, \cdots, f_n)),$$

where the f_i are continuous and satisfy a (local) Lipschitz-condition for (x, t) in some region $\Omega \times I$. Here Ω is assumed to be an open region in Euclidean n-space R^n, and I is the unit interval $0 \leq t \leq 1$, notation which we shall keep throughout this paper. A solution $\xi(t) = (\xi_1(t), \cdots, \xi_n(t))$ of the differential equations is called *periodic*, if it satisfies the boundary conditions $\xi_i(0) = \xi_i(1)$ $(i = 1, \cdots, n)$. In the theorem that is formulated below, we give conditions in terms of the functions f_i that guarantee the existence of periodic solutions in subregions of $\Omega \times I$. In order to make the nature of these conditions clearer we introduce them here in a more geometric way. Let us assign to each solution of the differential equation the curve in R^{n+1} with the parametric representation $(\xi_1(t), \cdots, \xi_n(t), t)$ and with the orientation given by increasing t. Let Z be a subregion of $\Omega \times I$, and let T be a sufficiently smooth hypersurface that belongs to the boundary of Z. We shall say that T is of uniform type with respect to Z if there are no two solution curves which intersect with T in such a way that one curve arrives from the interior and the other arrives from the exterior of Z. In the notation of Ważewski, this means T does not contain *points de sortie* (points of egress) as well as *points d'entrée* (points of ingress); see [3, p. 280], [1, p. 179].

We now consider a region $Z \subseteq \Omega \times I$ that is bounded by cylindrical and plane hypersurfaces S_0, S_1, T_i and T_i^* $(i = 1, \cdots, n)$. The hypersurface S_0 is

$$\{(x, t): t = 0,\ \alpha_i \leq x_i \leq \beta_i\},$$

and S_1 is

$$\{(x, t): t = 1,\ \alpha_i \leq x_i \leq \beta_i\}, \quad \text{where } \alpha_i = \alpha_i(0) = \alpha_i(1),\ \beta_i = \beta_i(0) = \beta_i(1).$$

The hypersurfaces T_i, T_i^* are defined by equations of the form $x_i = \alpha_i(t)$, $x_i = \beta_i(t)$, respectively, with $\alpha_i(t) \leq \beta_i(t)$ and $\alpha_i(0) = \alpha_i(1)$, $\beta_i(0) = \beta_i(1)$ $(i = 1, \cdots, n)$. Our theorem can be stated as follows: *If* $T_i \cup T_i^*$ *is of uniform type* $(i = 1, \cdots, n)$, *then there exists a periodic solution of the system* $\dot{x} = f(x, t)$ *inside* Z. It should be noted that if all T_i, T_i^* are of the same uniform type, then our statement is an immediate consequence of Brouwer's fixed point theorem. If, for example, there are no points of egress on $\bigcup_{i=1}^{n} T_i \cup T_i^*$, any solution that starts at some point $(x, 0) \in S_0$ cannot leave Z except at some point $(x, \bar{x}) \in S_1$. The mapping $x \to \bar{x}$ is

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