CELLULARITY OF SETS IN PRODUCTS

M. L. Curtis and D. R. McMillan

1. INTRODUCTION

There is no known factorization $R^n = X \times Y$ of euclidean n-space R^n in which neither factor is locally euclidean, although factorizations are known in which one factor fails to be locally euclidean (see [2] and [1]). There is a class of nonlocally euclidean spaces, which we call "pinched spaces" (see Section 5), and it seems likely that if X and Y are pinched spaces, then $X \times Y$ is euclidean space. We cannot show this, but, as a corollary to our main theorem, we have the conclusion that $X \times Y$ is a homotopy manifold.

The crucial question turns out to be whether certain sets are cellular (as defined by M. Brown [3]), and our main result is the following.

THEOREM 1. *Let* M^m *and* N^n *be combinatorial manifolds, and let* A *and* B *be absolute retracts in* Int M *and* Int N, *respectively. If* $\sup\{m\text{-dim } A, n\text{-dim } B\} \geq 2$, *then* $A \times B$ *is cellular in* $M \times N$. *In fact, if* $M \times N$ *is triangulated as a combinatorial manifold, then* $A \times B$ *is the intersection of combinatorial* $(m + n)$*-cells in* $M \times N$.

In the above context, $A \times B$ will be said to be *combinatorially cellular* in $M \times N$.

2. NESTED SEQUENCES OF MANIFOLDS

We collect here some results needed in proving Theorem 1.

(i) *Let* A *be an absolute retract in* Int M, *and let* U *be an open neighborhood of* A. *Then there exists a finite combinatorial manifold* H, *with nonempty boundary, such that*

$$A \subset \text{Int } H \subset H \subset U.$$

Such an H may be obtained as a small regular neighborhood of the closed simplicial neighborhood of A in a sufficiently fine subdivision of M.

(ii) *Let* $A \subset \text{Int } H$ *as in* (i). *Then there exists a neighborhood* V *of* A *such that* $V \subset \text{Int } H$ *and the inclusion* $i: V \to H$ *is null-homotopic.*

Since H is an absolute neighborhood retract, there exists an $\varepsilon > 0$ with the property that if f and g are maps of a space K into H such that $\rho(f(k), g(k)) < \varepsilon$ for each $k \in K$, then f and g are homotopic in H. Let r be a retraction of H onto A, and choose V to be an open set such that $A \subset V \subset \text{Int } H$ and $\rho(x, r(x)) < \varepsilon$ for each x in V. Since A is contractible, V is the required neighborhood of A.

(iii) *There exists a sequence* $\{H_i\}$ *of finite combinatorial* m*-manifolds, with nonempty boundaries, such that* $H_{i+1} \subset \text{Int } H_i$, $A = \bigcap_i H_i$, *and each inclusion* $H_{i+1} \to H_i$ *is homotopically trivial.* This follows immediately from (i) and (ii).

The following result is proved in [8].

Received May 17, 1962.