THE EQUATION $a^M = b^N c^P$ IN A FREE GROUP

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1. INTRODUCTION

The question of finding all solutions for the equation $a^M = b^N c^P$ in a free group is of interest only if none of the exponents is 0 or 1; we assume, then, that $M, N, P \geq 2$. The equation possesses obvious solutions for which a, b, and c are all powers of a common element; it will be shown that these are all solutions.

R. Vaught conjectured that $a^2 = b^2 c^2$ had only these obvious solutions, and R. C. Lyndon [3] verified this conjecture by a combinatorial argument. His result carries with it the case that all three exponents are even. That there are only the obvious solutions in the case where the three exponents have a common prime divisor was established independently by G. Baumslag [1], E. Schenkman [4], and J. Stallings [6], all of whom employed more characteristically group theoretic methods. The proof here, for general $M, N, P \geq 2$, is of a combinatorial nature.

In Section 2 we record some properties of the free monoid F of words representing elements in a free group G. In Section 3 we reduce the problem of finding all the solutions of the equation $a^M = b^N c^P$ in G to that of finding all solutions of each of two equations in F. In Sections 4 and 5 we show in turn that each of these equations has only the obvious solutions.

The greater part of the argument deals with the case that one of the exponents is 2 or 3. This suggests that arbitrary equations in powers of elements from a free group have only more or less obvious solutions when the exponents are sufficiently large. More generally, one may expect that in some sense more complicated equations have fewer solutions, with only rather special equations possessing genuinely nondegenerate solutions. Thus the equation $a^M = b^N c^P d^Q$, which possesses a wealth of nontrivial solutions when all four exponents are 2, appears to have only obvious solutions when all exponents are large.

2. COMBINATORIAL LEMMAS

Let G be a group freely generated by a set X of generators x. Let F be the monoid freely generated by the set $X \cup \bar{X}$, where $\bar{X}$ is a set, disjoint from X, of elements $\bar{x}$ in one-to-one correspondence with the elements x of X. The elements of F are *words*. A word a *represents* the group element ϕa, where ϕ is the epimorphism from F onto G carrying x into x and $\bar{x}$ into x^{-1}. The *length* $|a|$ of a word a is the number of factors in its expression as a product of the *letters* x and $\bar{x}$. The *formal inverse* $\bar{a}$ of a word a is its image under the involutory antiautomorphism of F that interchanges x and $\bar{x}$. Clearly $\phi\bar{a} = (\phi a)^{-1}$.

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