A SIMPLIFIED PROOF OF THE PARTITION FORMULA

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We give a short proof of the celebrated Hardy-Ramanujan formula

$$p(n) \sim \frac{\exp \pi \sqrt{2n/3}}{4n\sqrt{3}},$$

where p(n) represents the number of ways in which n can be written as a sum of positive integers (see [1, p. 79]).

This proof, like the old one, utilizes the circle method, with the difference that we have only one major arc and one minor arc. Another difference is that we do not use any results on theta functions or modular functions. Of course, our error term suffers as a result; but we do obtain the asymptotic formula.

Suppose, as usual, that we define $p(0) = 1$ and write

$$f(z) = \sum_{n=0}^{\infty} p(n) z^n = \prod_{m=1}^{\infty} (1 - z^m)^{-1} \qquad (|z| < 1).$$

We also write

$$\phi(z) = \left(\frac{1 - z}{2\pi}\right)^{1/2} \exp\left[\frac{\pi^2}{12}\left(-1 + \frac{2}{1 - z}\right)\right].$$

The crux of our proof is the establishment of the two estimates

(I) $$|f(z)| < \exp\left(\frac{1}{1 - |z|} + \frac{1}{|1 - z|}\right),$$

(II) $$f(z) = \phi(z)[1 + O(1 - z)] \qquad (|z| < 1,\ |1 - z| \leq 2(1 - |z|)).$$

Proof of (I). Taking logarithms, we obtain the well-known identity

(1) $$\log f(z) = \sum \frac{1}{n} \frac{z^n}{1 - z^n}.$$

Hence,

$$|\log f(z)| \leq \frac{|z|}{|1 - z|} + \sum_{n=2}^{\infty} \frac{1}{n} \frac{|z|^n}{1 - |z|^n}$$

$$< \frac{1}{|1 - z|} + \frac{1}{1 - |z|} \sum_{n=2}^{\infty} \frac{1}{n^2} \frac{n}{|z|^{-1} + |z|^{-2} + \cdots + |z|^{-n}}$$

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