ARITHMETICAL NOTES, VI. SIMULTANEOUS BINARY COMPOSITIONS INVOLVING COPRIME PAIRS OF INTEGERS

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1. INTRODUCTION

Let m, n denote positive integers, and let $Q \equiv Q(m, n)$ represent the number of sets of integers x_1, x_2, y_1, y_2 such that

$$(1.1) \qquad m = x_1 + y_1, \quad n = x_2 + y_2, \quad x_j > 0, \; y_j > 0 \qquad (j = 1, 2),$$

subject to the restriction

$$(1.2) \qquad (x_1, x_2) = (y_1, y_2) = 1.$$

It is the object of this note to prove

THEOREM A. *If* $n \geq m$, *then*

$$(1.3) \qquad Q(m, n) \sim mn\alpha(m, n) \qquad \textit{as } m \to \infty,$$

where

$$(1.4) \qquad \alpha(m, n) = \prod_{p \mid (m,n)} \left(1 - \frac{1}{p^2}\right) \prod_{p \nmid (m,n)} \left(1 - \frac{2}{p^2}\right).$$

(Throughout this note, p stands for a prime.)

As a consequence of this result, one may obtain

THEOREM B. *There exist positive constants* A_1, A_2 *such that, when* m *and* n *are sufficiently large*,

$$A_1 < Q(m, n)/mn < A_2.$$

Theorem A is actually proved in a slightly stronger form (see Theorem 3.1). The proof is based on an elementary method similar to that employed by Mirsky in [3].

2. SOME LEMMAS

Let $\theta_p(m, n)$ denote the number of solutions (mod p) of

$$m \equiv x_1 + y_1, \; n \equiv x_2 + y_2 \pmod p, \qquad p \nmid (x_1, x_2), \; p \nmid (y_1, y_2).$$

The following result is the special case r = p of [1, (8.8), Corollary 18.1].

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