ON THE COEFFICIENTS OF CLOSE-TO-CONVEX FUNCTIONS

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1. Let K_α $(0 \leq \alpha \leq 1)$ be the class of all functions

(1) $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

analytic in $|z| < 1$ that satisfy $f'(z) \neq 0$ and

$$\int_{\theta_1}^{\theta_2} \frac{\partial}{\partial\theta} \arg[e^{i\theta} f'(re^{i\theta})]\, d\theta > -\pi\alpha$$

for all $\theta_1 < \theta_2$ and $0 \leq r < 1$. The class K_1 is the class of close-to-convex functions [4], and the classes K_α are subclasses of K_1. Hence all functions $f \in K_\alpha$ $(0 \leq \alpha \leq 1)$ are univalent. The class K_0 consists of the convex functions. A function of the form (1) belongs to K_α if and only if there exists a function

(2) $$g(z) = \sum_{n=1}^{\infty} b_n z^n,$$

starlike in $|z| < 1$, such that (see [4] and [12])

$$\left|\arg \frac{zf'(z)}{g(z)}\right| < \frac{\pi}{2}\alpha .$$

M. O. Reade [12] has proved that

(3) $$|a_n| \leq 1 - \alpha + n\alpha .$$

For $\alpha = 0$ and $\alpha = 1$, this reduces to the sharp inequalities

(4) $$|a_n| \leq 1 \qquad (f \in K_0)$$

and [11]

(5) $$|a_n| \leq n \qquad (f \in K_1).$$

For $n = 2$, inequality (3) is best possible for every α. On the other hand, it will be shown that

$$a_n = O(n^\alpha) \qquad (n \to \infty).$$

For $0 < \alpha < 1$ and large n, this estimate is better than (3). For a function f of boundary rotation not greater than $2\pi + \pi\alpha$ (which implies $f \in K_\alpha$), Rényi has proved that $|a_n| \leq n^\alpha$.

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