IMAGES OF CONVEX DOMAINS UNDER CONVEX CONFORMAL MAPPINGS

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Let $w = f(z)$ be a convex mapping of $|z| < 1$; that is, let $f(z)$ map $|z| < 1$ conformally and one-to-one onto a convex domain. Study [1] has provided that $f(z)$ maps every disk in $|z| < 1$ onto a convex domain.

A circle in $|z| \leq 1$ that touches $|z| = 1$ is called an *oricycle*. Through each pair of points in $|z| < 1$ there pass exactly two oricycles.

THEOREM. *A convex set* C *in* $|z| < 1$ *is mapped onto a convex set by every convex mapping of* $|z| < 1$ *if and only if, for each pair of points* z_1 *and* z_2 *in* C, *the arcs between* z_1 *and* z_2 *of the two oricycles through* z_1 *and* z_2 *also belong to* C.

Proof. 1. Let C be a convex set in $|z| < 1$ that has the property just stated, and let z_1 and z_2 be two points in C. Let Z_1 and Z_2 be the two oricycles passing through both z_1 and z_2. Let K_1 and K_2 be the closed interiors of Z_1 and Z_2, and let $K = K_1 \cap K_2$. Then the boundary of K belongs to C, and therefore $K \subset C$.

Let K_1^*, K_2^*, K^*, and C^* be the images of K_1, K_2, K, and C. By Study's theorem, the sets K_1^* and K_2^* are convex; hence, the set $K^* = K_1^* \cap K_2^*$ is also convex. Since $K \subset C$, we have $K^* \subset C^*$; and because K^* is convex, the segment $[f(z_1), f(z_2)]$ also belongs to C^*. Therefore C^* is convex.

2. Let the image C^* of C be convex for every convex mapping of $|z| < 1$. Let Z be an oricycle through the points z_1 and z_2 of C, and let Z touch the unit circle at z_0. The function

$$w = \frac{z_0 + z}{z_0 - z}$$

maps $|z| < 1$ onto the half-plane $\Re w > 0$. Since the oricycle Z passes through z_0, it is mapped onto a straight line Z^*. Because C^* is convex, the segment

$$[f(z_1), f(z_2)]$$

of Z^* belongs to C^*. Hence the arc of Z between z_1 and z_2 belongs to C.

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REFERENCE

1. E. Study, *Vorlesungen über ausgewählte Gegenstände der Geometrie*, II. Heft, Leipzig, 1913.

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