# THE TAYLOR COEFFICIENTS OF INNER FUNCTIONS

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## INTRODUCTION

The object of the present paper is to study bounds on the Taylor coefficients of a function $f(z)$ that is regular and bounded by 1 in $|z| < 1$ and has boundary values $f(e^{i\theta})$ of modulus 1 for almost all $\theta$. We call such a function an *inner function* (terminology introduced by Beurling [1]). Inner functions play an important role in the study of functions of class $H_p$ (see for example Privalov [6, p. 53], Zygmund [9, Vol. I, p. 271]), in certain approximation questions [1], and in the study of the invariant subspaces of the "shift operator" in $\ell_2$ [1]. It is known [6] that the most general inner function is the product of a Blaschke product and a function of the form

$$\exp\left(\int_0^{2\pi} \frac{z + e^{it}}{z - e^{it}}\, d\rho(t)\right),$$

where $\rho(t)$ is a positive measure singular with respect to Lebesgue measure. The set of Taylor coefficients of an inner function can also be described, without reference to analytic functions, as a solution of the infinite system of quadratic equations

$$\sum_{n=0}^{\infty} |a_n|^2 = 1,$$

$$\sum_{n=0}^{\infty} a_n \bar{a}_{n+k} = 0 \qquad (k = 1, 2, \cdots).$$

Qualitatively, our main results are these: the coefficients of an inner function that is not a finite Blaschke product cannot be $o(1/n)$, although they can be $O(1/n)$; and if the function does not vanish in $|z| < 1$, they are sometimes $O(n^{-3/4})$ and never $o(n^{-3/4})$.

## 1. COEFFICIENTS OF INNER FUNCTIONS

THEOREM 1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an inner function, and denote by* $A_n$ *the infinite matrix*

$$\begin{pmatrix} |a_n| & |a_{n+1}| & |a_{n+2}| & \cdots \\ |a_{n+1}| & |a_{n+2}| & |a_{n+3}| & \cdots \\ |a_{n+2}| & |a_{n+3}| & |a_{n+4}| & \cdots \\ \cdots & & & \end{pmatrix}.$$

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