A NEW APPROACH TO THE FIRST FUNDAMENTAL THEOREM ON VALUE DISTRIBUTION

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1. The fundamental theorems of R. Nevanlinna in the theory of functions meromorphic in the plane or unit circle [8] were stated in a slightly different form by Ahlfors [1] by means of a new technique. Using the methods of Ahlfors, the author extended the results to functions meromorphic in more general domains D [5]. Later, analogous though weaker results were derived for functions of pseudomeromorphic character in D [6]. These functions can be characterized as being quasiconformal in every compact subdomain of D and having continuous partial derivatives. In this connection, formulas of Ozaki, Ono, and Ozawa [9] were useful. They can be interpreted as an unintegrated form of the first fundamental theorem.

The aim of the present paper is to give a new proof of the unintegrated first fundamental theorem, more in the style of the Ahlfors methods, and to state it for a wider class of functions than the one considered by the original authors. Further, we give a somewhat better estimate of the remainder term occuring in the analogue of the integrated first fundamental theorem for pseudomeromorphic functions; this estimate is valid for a slightly wider class than that previously considered. The corresponding theorem for functions of meromorphic character is an easy corollary of a step in the deduction; but the new approach is of course more involved than the original concept used for this special case in [1] and [5].

2. Let D be an open domain in the z-plane ($z = x + iy$), and Γ its boundary. Let D be exhausted by open subdomains Δ_λ ($-\infty < \lambda < \lambda_0$) with boundaries $G_\lambda \subset D$, each consisting of a finite number of Jordan curves. We assume Δ_λ to be increasing with λ, so that $\Delta_\lambda \cup G_\lambda \subset \Delta_{\lambda'}$ for $\lambda < \lambda'$, and so that Δ_λ tends to D for $\lambda \nearrow \lambda_0$ and to some inner point O (which for convenience we take as $z = 0$) as $\lambda \searrow -\infty$.

3. Let the function

$$w(z) = u(x, y) + i\,v(x, y)$$

provide a mapping of D into the Riemann w-sphere W. We assume that

a) $w(z)$ is continuous (in the spherical sense);

b) there is no accumulation point of the roots of $w(z) = a$ in D, for any a; and

c) the mapping is sense-preserving in the following strong sense: Whenever a sufficiently small circle with center $z_0 \in D$ is described once in the positive sense, then $\arg[w(z) - w(z_0)]$ increases by $2\pi k$, where k is a positive integer (unless $w(z_0) = \infty$, in which case the obvious analogue on W has to be considered).

From c) it follows that k can be defined as the multiplicity of an a-point z_0. We see at once that *the principle of the argument* remains valid, so that for a domain Δ bounded by Jordan arcs $G \subset D$,

Received January 29, 1962.