FAILURE OF THE KRULL-SCHMIDT THEOREM FOR INTEGRAL REPRESENTATIONS

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1. The following notation will be used throughout:

G is a finite group of order g;

K is an algebraic number field;

R is the ring of all algebraic integers in K;

P is a prime ideal in R;

R_P is the P-adic valuation ring in $K = \{\alpha/\beta\colon \alpha, \beta \in R, \beta \notin P\}$;

K_P^* is the P-adic completion of K, and R_P^* the ring of P-adic integers in K_P^*;

$\tilde{R} = \bigcap_{P|g} R_P = \{\alpha/\beta\colon \alpha, \beta \in R, R\beta + Rg = R\}$.

Let RG denote the group ring of G with coefficients from R. By an RG-*module* we shall always mean a finitely-generated left RG-module which is R-torsion-free, and upon which the identity element of G acts as identity operator. Analogous definitions hold for $R_P G$-modules, KG-modules, and so forth.

THEOREM 1.1 (Krull-Schmidt). *In any decomposition of a* KG-*module* M *into a direct sum of indecomposable submodules, the indecomposable summands are uniquely determined by* M, *up to* KG-*isomorphism and order of occurrence.*

The standard proof (see, for example, Curtis and Reiner [2, p. 83]) shows that K may be replaced by any commutative ring whose ideals satisfy the descending chain condition.

In the present paper we wish to consider the validity of the Krull-Schmidt theorem for RG-modules. Let us observe at once that the theorem already fails when $G = \{1\}$ if R contains non-principal ideals. Let $J_1, \cdots, J_n$ be ideals of R, and let $\dotplus$ denote the external direct sum operation. As is well known,

$$J_1 \dotplus \cdots \dotplus J_n \cong R \dotplus \cdots \dotplus R \dotplus J_1 \cdots J_n,$$

where n - 1 R's occur on the right-hand side.

Returning to an arbitrary finite group G, we might reasonably hope that the non-principal ideals of R are the only source of counterexamples. To avoid the difficulties arising from them, we may work with $\tilde{R}G$-modules instead of RG-modules, where $\tilde{R}$ is the principal ideal ring defined above.

To each RG-module M there corresponds an $\tilde{R}G$-module, denoted by $\tilde{R}M$ and defined by

$$\tilde{R}M = \tilde{R} \otimes_R M.$$

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