A NOTE ON THE HOMOLOGY GROUPS OF RELATIONS

John Thomas

The purpose of this paper is to give a new proof of Dowker's theorem that "related" homology and cohomology theories are isomorphic. In [1], Dowker observed that a relation between two sets X and Y endows each with the structure of a simplicial complex, and he showed that the homology groups of X and Y derived from these simplicial structures are isomorphic. The isomorphism constructed by Dowker seems rather complicated, and it is not immediately apparent that it is natural (as must be verified for Dowker's main application) that the Čech and Vietoris homology theories and the Čech and Alexander cohomology theories are isomorphic, when they are based on the same family of coverings.

In this paper we exhibit an isomorphism that is somewhat simpler, and (granted a certain amount of algebraic machinery) more clearly natural. Roughly speaking, we proceed as follows. We consider the category of relation pairs and their maps, and define three homology functors on this category—the two that Dowker considered, and a third, which is defined in terms of a certain double complex; and we show that all three functors are naturally isomorphic.

Here is a brief sketch of the isomorphism between the Čech and Vietoris homology theories. Consider the category of pairs (X, α) in which X is a topological space and α is a cover of X taken from a particular directed family of coverings, say $\Omega(X)$. The maps $(f, \phi)\colon (X, \alpha) \to (Y, \beta)$ are pairs in which f is continuous and ϕ is a function. The relation on $X \times \alpha$ is the inclusion relation: x in X and O in α are related if $x \in O$. Let the homology groups induced by the two simplicial structures be denoted by ${}^1H(X, \alpha)$ and ${}^2H(X, \alpha)$. The maps induced by

$$(1, \phi)\colon (X, \alpha) \to (X, \beta),$$

where α refines β and ϕ is relation preserving, give inverse systems for each X, and as Dowker pointed out,

$$\operatorname*{lim\,inv}_{\alpha\in\Omega(X)} {}^1H(X, \alpha) \quad \text{and} \quad \operatorname*{lim\,inv}_{\alpha\in\Omega(X)} {}^2H(X, \alpha)$$

are the Vietoris and Čech groups, respectively. Since the functors ${}^1H$ and ${}^2H$ are isomorphic, they are still isomorphic when composed with the functor lim inv, whence the Čech and Vietoris homology theories are isomorphic.

1. DOUBLE COMPLEXES

In this section we recall some of the basic facts about double complexes. Let A be a principal ideal domain. A *double complex* K over A consists of a bigraded left A-module $K = \Sigma K_{p,q}$ (p, q integers), together with a pair of homogeneous endomorphisms

Received January 12, 1962.

The result of this paper is a minor part of my doctoral dissertation at the University of Oklahoma, June 1959. I wish to express my gratitude to Professor John Giever for his help and encouragement during the preparation of that thesis.