A NOTE ON ε-MAPS ONTO MANIFOLDS

Tudor Ganea

1. Let X be an n-dimensional, compact, metric absolute neighborhood retract and suppose that for every $\varepsilon > 0$ there exists a map f_ε of X onto a closed n-dimensional manifold Y_ε such that each of the sets $f_\varepsilon^{-1}(y)$, $y \in Y_\varepsilon$, has diameter less than ε. Then, as is shown in [4], X has the homotopy type of a closed n-dimensional manifold, its separation properties by closed subsets are similar to those of closed n-manifolds, and, if $n = 2$, X is homeomorphic to a closed surface. Therefore, it is natural to ask whether such a space X is necessarily homeomorphic to a manifold. The purpose of this note is to produce an example of a compact, metric, 3-dimensional absolute neighborhood retract that may be mapped onto the 3-sphere with arbitrarily small counter-images, but that fails to be a manifold.

2. Let X denote the quotient space obtained from the 3-sphere S^3 by shrinking the wild arc described in Example 1.1 of [3] to a point; let $\phi: S^3 \to X$ denote the identification map, and let $a = \phi(A)$, where A denotes the wild arc. Clearly, X is a 3-dimensional, compact, metrizable space, and, according to the Borsuk-Whitehead theorem [1], [5], it is an absolute neighborhood retract. That X is not a manifold, since it fails to be locally Euclidean at the point a, has already been pointed out in [2, p. 156].

Let d be any distance function in X, and note that for a compact space, the property of admitting maps onto S^3 with arbitrarily small counter-images does not depend on the particular choice of d. Now, let $\varepsilon > 0$ be given. The continuity of ϕ yields an $\eta > 0$ such that any subset $E \subset S^3$ satisfies the inequality

$$\text{(1)} \qquad \operatorname{diam} \phi(E) < \varepsilon/2 \quad \text{if} \quad \operatorname{diam} E < \eta .$$

Let U and V be open cubical neighborhoods of the end points p and q of A such that $\overline{U} \cap \overline{V} = \emptyset$ and

$$\text{(2)} \qquad \operatorname{diam} U < \eta, \qquad \operatorname{diam} V < \eta .$$

Select points r and s on $A - (p \cup q)$ such that the subarcs B and C of A, joining p with r and s with q, be entirely contained in U and V, respectively. Also, define maps

$$h': (A \cap \overline{U}) \cup (\overline{U} - U) \to \overline{U} \quad \text{and} \quad k': (A \cap \overline{V}) \cup (\overline{V} - V) \to \overline{V}$$

by

$$h'(x) = r \text{ if } x \in B, \quad h'(x) = x \text{ otherwise},$$

$$k'(x) = s \text{ if } x \in C, \quad k'(x) = x \text{ otherwise}.$$

Since the closed cubes $\overline{U}$ and $\overline{V}$ are absolute retracts, we may extend the maps h' and k' to maps

Received November 30, 1961.