ISOMETRIC IMMERSION OF FLAT RIEMANNIAN MANIFOLDS IN EUCLIDEAN SPACE

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1. INTRODUCTION

Let ψ: M $\to \overline{M}$ be an isometric immersion of Riemannian manifolds. If z is a tangent vector of $\overline{M}$, orthogonal to $d\psi(M_m)$, there is a classically defined second fundamental form operator S_z on the tangent space M_m. Following [1], we express the same information about ψ by associating with each vector $x \in M_m$ a linear operator T_x on $\overline{M}_{\psi(m)}$, called the *difference operator* of x. The function T is characterized by the fact that each T_x is skew-symmetric and $T_x(z) = d\psi(S_z(x))$ for $x \in M_m$, where z has the same meaning as above. The symmetry of S_z is equivalent to the relation $T_x(d\psi(y)) = T_y(d\psi(x))$ for x, y $\in M_m$. If $m \in M$, let $\mathcal{N}(m)$ be the subspace of M_m consisting of all vectors x such that $T_x = 0$, and let $\nu(m)$ be the dimension of $\mathcal{N}(m)$. Chern and Kuiper [2] call this integer the *index of relative nullity of* ψ *at* m. We denote by n the minimum value of the function ν on M. Finally let $\mathcal{N}^{\perp}(m)$ be the orthogonal complement of $\mathcal{N}(m)$ in M_m.

We shall deal with the immersion ψ: $M^d \to R^{d+k}$ of a flat d-dimensional Riemannian manifold in (d + k)-dimensional Euclidean space. In this case the proof of Theorem 2 of [2] implies that *for each point* $m \in M$ *there exists a vector* $x \in \mathcal{N}^{\perp}(m)$ *such that* T_x *is one-to-one on* $d\psi(\mathcal{N}^{\perp}(m))$. Since the latter subspace has dimension d - $\nu(m)$, it follows that $k \geq d - \nu(m)$, so that the minimum relative nullity n of ψ is at least d - k. We shall prove

THEOREM 1. *Let* ψ: $M^d \to R^{d+k}$ *be an isometric immersion of a complete flat Riemannian manifold in Euclidean space. Then* M^d *contains a totally geodesic submanifold that is carried isometrically onto an entire* n*-dimensional plane in* R^{d+k}, *where* n *is the minimum relative nullity of* ψ.

The theorem is trivially true if n is zero, but since $n \geq d - k$ we can force n to be positive:

COROLLARY 1. *If the hypotheses of Theorem* 1 *are satisfied and* $k < d$, *then the image of* ψ *contains a* (d - k)*-dimensional plane in* R^{d+k}.

This implies the fundamental result of Tompkins [4] that a compact flat M^d cannot be isometrically immersed in R^{2d-1}. More generally, we have

COROLLARY 2. *A complete flat Riemannian manifold* M^d *does not have a bounded isometric immersion in* R^{2d-1}.

As with Tompkins' theorem, restrictions on dimension cannot here be weakened, for R^d has bounded imbeddings in R^{2d}, indeed, imbeddings whose images are as small as one likes: imbed R^1 as, say, a small spiral in R^2, then take the d-fold Riemannian product.

For k = 1, that is, for the case of a hypersurface, Hartman and Nirenberg have proved (Theorem III of [3]) that an isometric immersion of a complete flat M^d in R^{d+1} is cylindrical. In Theorem 2 we give a sufficient condition for such immersions to be cylindrical when $k > 1$.

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