A THEOREM ON MAPS WITH NON-NEGATIVE JACOBIANS

Albert Nijenhuis and R. W. Richardson, Jr.

In this note we give an elementary proof of the following theorem:

THEOREM 1. *Let* M *and* N *be connected, oriented* n*-dimensional differentiable manifolds of class* C^k $(k \geq 1)$, *and let* $f: M \to N$ *be a proper* C^k *map such that the Jacobian* J(f) *is non-negative at each point of* M. *Then either* $J(f) \equiv 0$ *or* f *maps* M *onto* N.

We say that f is *proper* if $f^{-1}(C)$ is compact for every compact subset C of N. We note that since M and N are oriented, the sign of J(f) is well defined at each point of M.

Theorem 1 was proved for compact M in [2]. The proof of [2] uses the homological ideas of degree and local degree and requires the application of Sard's theorem on critical values. Our proof is motivated by de Rham's theorem, but makes no explicit use of the de Rham cohomology. It is self-contained, makes no reference to homology theory, and uses only elementary facts from the calculus of differential forms and, in the C^1 case, the generalized Stokes' formula.

1. AN ELEMENTARY CASE OF GREEN'S THEOREM

The material in this section is known [1], but we include it for the sake of completeness. The following lemma is a weak form of Green's theorem in R^n. We refer the reader to [1] for the standard facts concerning differential forms.

LEMMA 1. *Let* ω *be a differential* (n - 1)*-form of class* C^1 *in* R^n *with compact support. Then* $\int_{R^n} d\omega = 0$.

Proof. In a coordinate system $(x_1, \cdots, x_n)$,

$$\omega = \sum_{i=1}^{n} a_i(x_1, \cdots, x_n)\, dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n .$$

Thus

$$d\omega = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial a_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n .$$

Choose a number $r > 0$ such that $\omega \equiv 0$ outside of the set

$$\{(x_1, \cdots, x_n) \mid |x_i| < r, \ i = 1, \cdots, n\} .$$

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