UNIQUENESS THEOREMS FOR ORDINARY AND HYPERBOLIC DIFFERENTIAL EQUATIONS

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1. This note is concerned with uniqueness theorems for the ordinary differential equation $x' = f(t, x)$ and for the hyperbolic partial differential equation

$$u_{xy} = f(x, y, u, p, q).$$

We first consider a uniqueness result for the ordinary differential equation, under assumptions more general than the uniqueness hypothesis of Kamke [2, pp. 48-52], and then extend this result to prove the analogous uniqueness theorem for the hyperbolic partial differential equation.

Uniqueness theorems for the partial differential equation, under what may be called Nagumo's uniqueness conditions, have been considered by Diaz and Walter [3] and Shanahan [4]. A "crude analogue" of Kamke's uniqueness assertion also appears in [4].

2. Consider the ordinary differential equation

$$x' = f(t, x), \qquad x(0) = 0, \tag{1}$$

where $f(t, x)$ is a real-valued function defined on $0 \leq t \leq a$, $|x| \leq b$. A solution of (1) in the classical sense will mean a real-valued function $x(t)$, continuous in $0 \leq t \leq a$ and having a finite derivative $x'(t)$, for $0 < t < a$, that satisfies $x'(t) = f(t, x(t))$ for $0 < t < a$. Suppose $x(t)$ and $y(t)$ are solutions of (1) existing on $0 \leq t \leq a$; then the requirement

$$\lim_{t \to 0+} \frac{|x(t) - y(t)|}{t} = 0,$$

which is satisfied when f is continuous at (0, 0), is a necessary condition for the uniqueness of solutions. This requirement can be generalized. Suppose

$$\lim_{t \to 0+} \frac{|x(t) - y(t)|}{B(t)} = 0, \tag{2}$$

where the function $B(t)$ is continuous, positive on $0 < t \leq a$ and such that $B(0+) = 0$. This condition is necessary for uniqueness, but not sufficient. As a matter of fact, we prove

LEMMA 1. *Suppose the function* $B(t)$ *is continuous, positive on* $0 < t \leq a$, *with* $B(0+) = 0$. *Then there exists an infinity of functions* f *such that* (1) *has more than one solution satisfying the condition* (2).

Proof. We first construct a function $A(t)$ having a non-negative derivative on $0 \leq t \leq a$ and such that $\lim_{t \to 0+} A(t)/B(t) = 0$. We proceed as follows:

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