ON ASYMMETRIC DIOPHANTINE APPROXIMATIONS

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Our purpose is to give a brief proof of the following theorem of B. Segre [5].

Let τ be any non-negative real number. Every irrational number θ has infinitely many rational approximations h/k satisfying

(1) $$-\frac{1}{(1+4\tau)^{1/2}k^2} < \theta - \frac{h}{k} < \frac{\tau}{(1+4\tau)^{1/2}k^2}.$$

Segre's proof was geometric in nature. C. D. Olds [3] gave a proof using Farey sequences for the cases $\tau > 1$. Proofs by continued fractions have been given by N. Negoescu [2] and R. M. Robinson [4]. W. J. LeVeque [1] showed that (1) holds for at least one of any five consecutive convergents of the continued fraction expansion of θ. We give a short proof of Segre's theorem, using Farey sequences.

LEMMA. *Let θ be an irrational number, and let τ be any nonnegative real number. Let* a/b *and* c/d *be the two consecutive fractions of the Farey series* F_n *between which θ lies, and suppose that*

(2) $$\frac{a}{b} < \frac{a+c}{b+d} < \theta < \frac{c}{d}.$$

Then (1) *holds with* h/k *replaced by at least one of* a/b, (a + c)/(b + d), *and* c/d.

Proof. Define λ and μ by

$$\lambda = (1+4\tau)^{-1/2} \qquad \text{and} \qquad \mu = \tau(1+4\tau)^{-1/2},$$

so that $\mu = (1-\lambda^2)/4\lambda$ and $0 < \lambda \le 1$. Assuming that the conclusion of the lemma is false, we can write

(3) $$\theta - \frac{a}{b} \ge \frac{\mu}{b^2}, \quad \theta - \frac{a+c}{b+d} \ge \frac{\mu}{(b+d)^2}, \quad \frac{c}{d} - \theta \ge \frac{\lambda}{d^2}.$$

Adding the first and third of these inequalities, and also the second and third, we obtain the relations

$$\frac{c}{d} - \frac{a}{b} = \frac{1}{bd} \ge \frac{\mu}{b^2} + \frac{\lambda}{d^2},$$

$$\frac{c}{d} - \frac{a+c}{b+d} = \frac{1}{d(b+d)} \ge \frac{\mu}{(b+d)^2} + \frac{\lambda}{d^2},$$

in other words,

(4) $$\lambda b^2 - bd + \mu d^2 \le 0, \quad \lambda(b+d)^2 - d(b+d) + \mu d^2 \le 0.$$

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