ISOMORPHISMS OF THE ENDOMORPHISM RING OF A FREE MODULE OVER A PRINCIPAL LEFT IDEAL DOMAIN

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1. INTRODUCTION

The problem considered here is a special case of the general problem of the extent to which a module (F, A) is determined by its endomorphism ring E(F, A). We consider the special case of free modules A, over principal left ideal domains F (not necessarily commutative rings without proper zero divisors, in which each left ideal is principal). Let (F, A) and (G, B) be two such modules (of arbitrary rank). We are able to show (Theorem 4.1) that any isomorphism of E(F, A) and E(G, B) is induced by a semilinear transformation of (F, A) upon (G, B). The theorem is proved by modifying the methods used in [3] by Baer, who studied the problem when F and G are division rings. Use is made of the results in [9] on the lattice of submodules of free modules, and of the fact that the subring $E_0(F, A)$ of endomorphisms of finite rank determines the behavior of the entire ring E(F, A).

Other studies of the isomorphisms of endomorphism rings of modules appear in [1], [2], and [7]. The modules considered in these papers are (essentially) torsion modules A over (commutative) complete discrete valuation rings F. These rings are far more restricted than our rings of scalars. Our modules are torsion-free, and they are restricted by the requirement of the existence of a basis.

A related problem for groups is considered in [8]. The modules (F, A) studied there are free, of finite rank, over principal ideal domains in which each left and right ideal is principal. The authors determine all automorphisms of the unit group of E(F, A), in case A has rank at least three over F.

2. DEFINITIONS AND PRELIMINARIES

Throughout the paper, F and G will denote rings with identities, and (F, A) will indicate a unitary left F-module A. The set of all linear functionals on A (F-homomorphisms of A into F) forms a right F-module called the *adjoint module*, and it is denoted by (A*, F). We shall denote by E(F, A) the ring of all F-endomorphisms of A. The elements of E(F, A) shall operate on the elements of A from the right. If $x \in A$ and $y \in A^*$, the effect of the homomorphism y on the element x will be denoted by (x, y).

The ring F (not necessarily commutative) will be called a *principal left ideal domain* if it is a ring without proper zero divisors, and if moreover each left ideal is a principal left ideal. Any torsion-free module (F, A) over such a ring may be assigned a unique *rank* r(A), which is the cardinal number of any of its maximal linearly independent subsets. The proof of this fact for torsion-free groups in [5, pp. 29-33] applies here, if one replaces the arguments involving commutativity of

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