ON THE ALGEBRAIC CLOSURE OF A PLANE SET

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1. Let Z be a set of complex numbers. Where convenient, Z will be identified in the natural way with the corresponding subset of the Euclidean plane E. We shall always assume that $\{-1, 0, 1\} \subset Z$; otherwise, Z is arbitrary. Define $Z_1 = Z$,

$$Z_{n+1} = \left\{ z: \sum_{j=0}^{N} a_j z^j = 0, \ \{a_0, \cdots, a_N\} \subset Z_N, N \geq 1, \ a_N \neq 0 \right\} \quad (n = 1, 2, \cdots).$$

Then $Z_1, Z_2, \cdots$ is an ascending sequence of sets, and we call the set

$$Z_\omega = \lim_{n \to \infty} Z_n = \bigcup_{n=1}^{\infty} Z_n$$

the *algebraic closure* of Z. It is clearly the smallest set that contains Z and is algebraically closed in the usual sense. In this note we shall study some properties of the sets Z_ω and of the relation of Z_ω to Z. Thus, we consider, in a sense, the algebraic closure apart from any algebraic structure.

If A denotes a set, an A-*equation* is an algebraic equation in a single unknown and with all coefficients in A. Any topological terms refer to the usual topology of E, and any group-theoretic ones to the ordinary multiplication of complex numbers.

2. LEMMA 1. *$Z_\omega - \{0\}$ is an Abelian group containing for every positive integer n every n-th root of every one of its elements.*

This follows directly from the definition of Z_ω and from the observation that if $z_1, z_2 \in Z_\omega$, then the equations $z_1 z - 1 = 0$, $z + z_2 = 0$, $z/z_1 - z_2 = 0$, and $z^n - z_1 = 0$ are Z_ω-equations.

THEOREM 1. *Z_ω is dense in E.*

By Lemma 1, Z_ω contains the group U of all roots of unity. It is easily verified that Z_ω contains two positive numbers, a and b, such that $\log a/\log b$ is irrational. For instance, $z^2 - z - 1 = 0$ is a Z_ω-equation, so that $a = (1 + 5^{1/2})/2 \in Z_\omega$; also, $z^2 - z - a = 0$ is a Z_ω-equation, and therefore $b = [1 + (3 + 2 \cdot 5^{1/2})^{1/2}]/2 \in Z_\omega$. Suppose now that $\log a/\log b$ is rational. Then $a^p = b^q$ for some positive integers p and q, which implies that $(3 + 2 \cdot 5^{1/2})^{1/2} = r + 5^{1/2} s$, with r and s rational. This implies further that $r^2 - 3rs + 5s^2 = 0$, which leads to $r = s = 0$, and this is a contradiction.

It follows that the module $\{n \log a + m \log b\}$, where n and m range independently over the rational integers, contains numbers of arbitrarily small absolute value, and hence is dense on the real line. Therefore the group S, generated by a

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