SOLUTION OF A GEOMETRIC PROBLEM BY FEJES TÒTH

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By "distance between two points on the unit sphere" or, in short, "distance between two points," we mean the length of the smaller of the two arcs of the great circle passing through the two points, and for two points that are diametrically opposite, one half of the circumference of a great circle. In [1], Fejes Tòth conjectured that the sum of the $\binom{s}{2}$ distances between s points on the unit sphere is at most $s^2\pi/4$ if s is even and at most $(s^2 - 1)\pi/4$ if s is odd. In the same paper, the author proved the conjecture for $s \leq 6$. The conjecture for even s was proved by Sperling in [2]. In this paper we shall show that the conjecture is true for odd s.

The first part of this paper is parallel to what Sperling did in [2], and we shall closely adhere to his notation. In the remainder of this paper, s is a positive odd integer. If $P_1, P_2, \cdots, P_s$ are s arbitrary points on the unit sphere (not necessarily distinct), we denote the distance between P_i and P_j by $\widehat{P_i P_j}$, and we put

$$E = \frac{1}{2} \sum_{i,j=1}^{s} \widehat{P_i P_j}.$$

If P_i' is the point diametrically opposite P_i, then one half of the sum of the $\widehat{P_i' P_j'}$ is E. Observing that the sum of the distances of any point from any two diametrically opposite points is π, we readily see that for the sum of the $\binom{2s}{2}$ distances between the $2s$ points $P_1, \cdots, P_s, P_1', \cdots, P_s'$ we have the equality

$$2E + \sum_{i,j=1}^{s} \widehat{P_i P_j'} = s^2\pi. \tag{1}$$

We denote the unit vectors from the center of the sphere to the points P_i and P_i' by x_i and x_i', respectively. For the inner product of the two vectors x_i and x_j we write (x_i, x_j). Henceforward, by "vector" we mean "unit vector."

If the angle α has the meaning indicated in Diagram 1, in which the diameters AA' and $P_i P_i'$ are perpendicular, then $\alpha = \arcsin(x_i, x_j)$. From Diagram 1 we see that

$$\widehat{P_i P_j'} = \widehat{P_i P_j} + 2\alpha = \widehat{P_i P_j} + 2 \arcsin(x_i, x_j),$$

which in turn implies that

$$\sum_{i,j=1}^{s} \widehat{P_i P_j'} = \sum_{i,j=1}^{s} \widehat{P_i P_j} + 2 \sum_{i,j=1}^{s} \arcsin(x_i, x_j) = 2E + 2 \sum_{i,j=1}^{s} \arcsin(x_i, x_j). \tag{2}$$

Received July 11, 1961.