ON A CONJECTURE OF GOODMAN CONCERNING MEROMORPHIC UNIVALENT FUNCTIONS

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1. Several authors have studied functions meromorphic and univalent in the unit circle, and some of them have made the further specialization of requiring the functions to possess a (simple) pole at a specified point of that circle [3, 4, 5, 6]. In addition, Goodman [1] has studied the class of functions that are *meromorphic and typically real* in the unit circle. Let such a function $f(z)$ have a pole at the point p $(0 < p < 1)$, and expansion about the origin given by

$$f(z) = z + \sum_{n=2}^{\infty} B_n z^n; \tag{1}$$

also, let

$$B(n, p) = (1 + p^2 + \cdots + p^{2n-2})/p^{n-1}.$$

It follows as a special case of a result of Goodman that

$$B_n \leq B(n, p).$$

Goodman denoted by $U(p)$ the class of functions $f(z)$, *meromorphic and univalent* in $|z| < 1$, with a pole at $z = p$ $(0 < p < 1)$ and with an expansion of the form (1) about the origin. In view of the preceding result, he made the conjecture (analogous to the Bieberbach conjecture) that the inequality $|B_n| \leq B(n, p)$ holds for each function in $U(p)$. Komatu [4] proved the inequality for the case $n = 2$.

The purpose of the present paper is to show that the conjecture is an easy consequence of the Bieberbach conjecture. In fact it is valid up to any stage for which the Bieberbach conjecture is true, so that in particular it holds at least for $n = 3$ and 4.

2. Let us denote by $E(p)$ the domain obtained from the unit circle $|z| < 1$ by deleting the segment $p \leq z < 1$, where $0 < p < 1$. Let us denote by $S(p)$ the class of functions regular and univalent in $E(p)$ with expansion of the form (1) about the origin. Then $S(1)$ is the class we usually denote by S [2, p. 1]. The domain $E(p)$ can be mapped conformally onto the unit circle by a function $k(z)$ with expansion about the origin

$$k(z) = (1 + p)^2 (4p)^{-1} z + \sum_{n=2}^{\infty} c_n z^n.$$

It is readily verified, for example from Löwner's parametric representation, see [7], that $c_n > 0$ for all n. Indeed, in equation (63), the function $\kappa(\tau) \equiv 1$. Now we

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