HOLOMORPHIC FUNCTIONS, OF ARBITRARILY SLOW GROWTH, WITHOUT RADIAL LIMITS

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By the well-known theorem of Fatou, if f(z) is holomorphic and bounded in $|z| < 1$ then f(z) possesses radial limits almost everywhere. This result was extended by Nevanlinna to meromorphic functions of bounded characteristic T(r) [4, p. 189]. A natural question raised by Lohwater and Piranian [2, p. 16] is this: if the condition of boundedness of T(r) be relaxed to the requirement that $T(r) < q(r)$, where $q(r) \to \infty$ slowly enough, can one still conclude that *some* radial limits must exist? Bagemihl, Erdös and Seidel [1, Theorem 7] have given an example of a *holomorphic* function without a radial limit for which $T(r) = O((1 - r)^{-8})$. Lohwater and Piranian [2] gave an example of a *meromorphic* function without radial limit for which $T(r) = O(-\log(1 - r))$. See also Noshiro [5, p. 90]. Mac Lane [3] gave an example of a *meromorphic* function, of arbitrarily slow growth, without asymptotic value (and hence without radial limit). The purpose of the present note is to derive a similar result for *holomorphic* functions. The method of proof and the precise statement of the result are different in the holomorphic case, since a holomorphic function must possess at least one asymptotic value (along some curve, not necessarily along some radius). For that reason the construction used in our example for meromorphic functions is completely inapplicable.

Let C_{-1} and C_1 be two fixed disjoint compact simple arcs in $|\zeta| < 1$, neither of which contains the origin, and such that each radius of $|\zeta| < 1$ intersects both C_{-1} and C_1. For example, we may use the two arcs $2\pi \leq \arg \zeta \leq 4\pi$ and $6\pi \leq \arg \zeta \leq 8\pi$ of the spiral $|\zeta| = 1 - (\arg \zeta)^{-1}$.

LEMMA 1. *There exists a function* $\phi(\zeta)$, *holomorphic in* $|\zeta| < 1$, *and a constant* $M > 1$ *such that*

$$|\phi(\zeta)| \leq M|\zeta| \qquad (|\zeta| < 1) \tag{1}$$

and

$$\begin{cases} \Re\, \phi(\zeta) \leq -1 & (\zeta \in C_{-1}), \\ \Re\, \phi(\zeta) \geq 1 & (\zeta \in C_1). \end{cases} \tag{2}$$

Proof. The three sets $\{0\}$, C_{-1} and C_1 may be enclosed in simply-connected neighborhoods, D_0, D_{-1}, D_1, whose closures are disjoint. Define the function $\phi_0(\zeta)$, holomorphic in $D_0 \cup D_{-1} \cup D_1$, by

$$\phi_0(\zeta) = 0 \;\; (\zeta \in D_0), \qquad \phi_0(\zeta) = -3 \;\; (\zeta \in D_{-1}), \qquad \phi_0(\zeta) = 3 \;\; (\zeta \in D_1).$$

Then, by Runge's theorem (see for example [6, p. 15]), there exists a polynomial $P(\zeta)$ approximating $\phi_0(\zeta)$ well enough so that

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