TYPES OF AMBIGUOUS BEHAVIOR OF ANALYTIC FUNCTIONS

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Let $w = f(z)$ be a complex-valued function, defined on the open disk D composed of all complex numbers z such that $|z| < 1$, and let z_0 be a point on the unit circle C, that is, on the boundary of D. We recall three definitions:

(1) The *cluster set* of f at z_0, $C(f, z_0)$, is the set of all points w on the Riemann sphere such that for each open set U containing z_0, every open set containing w meets the set $f(U \cap D)$. That is, w is in $C(f, z_0)$ provided there exists a sequence $\{z_j\}$ in D such that $z_j \to z_0$ and $f(z_j) \to w$.

(2) The *boundary cluster set* of f at z_0, $C_B(f, z_0)$, is the set of all points w such that for each open set U containing z_0, every open set containing w meets $\bigcup C(f, z')$, where the union is taken over all z' in $U \cap (C - z_0)$. If f is continuous, $C(f, z_0)$ is connected, but $C_B(f, z_0)$ need not be; however, if $C_B(f, z_0)$ is not connected, it has exactly two components, and these coincide with the right and left boundary cluster sets (definition obvious!), respectively. A semi-classical theorem of Iversen asserts that if f is meromorphic, then the boundary of $C(f, z_0)$ is contained in $C_B(f, z_0)$. (See, for example, [6, Theorem 1'].)

(3) For an *arc* A *terminating at* z_0 on C (we shall always mean, by this expression, an arc lying in D except for one endpoint at z_0), the *arc-cluster set* of f on A, $C(f, A, z_0)$, is the set of all points w such that for every open set U containing z_0, every open set containing w meets $f(A \cap U)$. Each arc-cluster set is connected, if f is continuous.

It may happen, even with bounded analytic functions, that at a point z_0 in C there exist two arcs, A_1 and A_2, terminating at z_0, for which the sets $C(f, A_1, z_0)$ and $C(f, A_2, z_0)$ are disjoint. If this does occur at z_0, we shall say that z_0 is a *point of disjoint cluster sets* [10]. Bagemihl has shown [1] that even for a purely arbitrary function the set of points of disjoint cluster sets is at most countable. The restriction of the discussion to analytic functions does not strengthen the conclusion. In fact, as Gross [7, Section 8] has shown, corresponding to each countable set $X = \{x_n\}$ on C, we can choose positive numbers $\{a_n\}$ such that $\{x_n\}$ is precisely the set of points of disjoint cluster sets of the bounded function

$$\exp\left(\sum_{n=1}^{\infty} a_n \cdot \frac{z_n + x_n}{z_n - x_n}\right).$$

If the condition of boundedness is replaced by the weaker condition that f be of bounded characteristic, it is even possible (see [3] and [9]) to require that for every x_n in X, the disjoint sets $C(f, A_1, x_n)$ and $C(f, A_2, x_n)$ each consist of one point (a point x_n exhibiting this phenomenon is called an *ambiguous point*).

Lohwater pointed out in [8] that the property of being a point of disjoint cluster sets is a special case of a more general property, and he proposed the investigation

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