MEROMORPHIC CLOSE-TO-CONVEX FUNCTIONS

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1. INTRODUCTION

Kaplan [4] has called a function f(z), analytic in $|z| < 1$, close-to-convex in $|z| < 1$ if there exists an analytic function $\phi(z)$, convex and schlicht in $|z| < 1$, such that

$$\Re \frac{f'(z)}{\phi'(z)} > 0 \qquad (|z| < 1). \tag{1.1}$$

Since $F(z) = z\phi'(z)$ is star-like with respect to the origin and schlicht in $|z| < 1$, (1.1) may be written in the alternative form

$$\Re \frac{zf'(z)}{F(z)} > 0 \qquad (|z| < 1). \tag{1.1'}$$

Kaplan has shown [4] that every close-to-convex function f(z) is schlicht in $|z| < 1$, and that, under the hypothesis that f'(z) does not vanish in $|z| < 1$, the condition (1.1) is equivalent to the alternative condition

$$\int_{\theta_1}^{\theta_2} \Re \left(1 + re^{i\theta} \frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right) d\theta > -\pi \tag{1.2}$$

for $\theta_1 < \theta_2$, $0 < r < 1$. The geometric interpretation of (1.2) is that $w = f(z)$ maps each circle $|z| = r < 1$ onto a simple closed curve whose tangent rotates, as θ increases, in such a way that it never turns back on itself sufficiently in the clockwise direction to reverse its direction completely.

In this paper we extend the concept of close-to-convex functions to meromorphic functions

$$f(z) = \frac{1}{z} + a_0 + a_1 z + \cdots + a_n z^n + \cdots \tag{1.3}$$

regular in $0 < |z| < 1$ and with a simple pole at the origin. We say that f(z), when given by (1.3), is close-to-convex in the punctured circle $0 < |z| < 1$ relative to F(z) if there exists a meromorphic, star-like, schlicht function F(z) in $|z| < 1$ with a simple pole at the origin, given by

$$F(z) = \frac{b_{-1}}{z} + b_0 + b_1 z + \cdots + b_n z^n + \cdots \qquad (b_{-1} \neq 0), \tag{1.4}$$

such that

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