APPROXIMATION OF ALGEBRAIC NUMBERS BY ALGEBRAIC NUMBERS

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Let $P(x) = \Sigma_{i=0}^{n} a_i x^i$ be a polynomial with arbitrary complex coefficients whose leading coefficient a_n is not 0. We call n the *degree*, $h = \max_{i \leq n} |a_i|$ the *height*, and $s = \Sigma_{i=0}^{n} |a_i|$ the *size* of the polynomial P(x). To every algebraic number α there corresponds a polynomial P(x) of lowest degree with $P(\alpha) = 0$ and such that its coefficients are rational integers without a common divisor. The degree, the height, and the size of this polynomial are called the *degree*, the *height*, and the *size* of α, respectively. We denote the set of all polynomials with rational integral coefficients whose degrees, heights, and sizes are $n > 0$, $h > 0$, and $s > 0$, respectively, by $\mathfrak{P}(n, h, s)$, and the set of all algebraic numbers satisfying the same conditions by $\mathfrak{A}(n, h, s)$. By $\mathfrak{P}^*(n, h, s)$ we denote the corresponding set of polynomials with arbitrary complex coefficients. In order to have a simple way of stating the theorems, we shall make use of these symbols even if not all of the numbers n, h, and s are actually needed.

It is well known that, for an algebraic number $\alpha \in \mathfrak{A}(m, h, s)$, the value of a polynomial $P(x) \in \mathfrak{P}(n, k, t)$ for which $P(\alpha) \neq 0$ cannot be arbitrarily small. In T. Schneider's *Einführung in die transzendenten Zahlen* we find the proof of the following theorem [10, Theorem 3]:

Let $\alpha \in \mathfrak{A}(m, h, s)$ *be an algebraic number whose leading coefficient is* a, *and let* $P(x) \in \mathfrak{P}(n, k, t)$ *be a polynomial for which* $P(\alpha) \neq 0$. *Then*

$$(1) \qquad |P(\alpha)| > |a|^{-nm} (n+1)^{-n(m-1)} (h+1)^{-n(m-1)} k^{-(m-1)}.$$

A similar theorem holds for polynomials in several variables. N. I. Feldman ([3], Lemma 6; [4], Lemma 2) proved the following result:

Let

$$A(x_1, \cdots, x_m) = \sum_{i_1=0}^{N_1} \sum_{i_2=0}^{N_2} \cdots \sum_{i_m=0}^{N_m} a_{i_1 i_2 \cdots i_m} x^{i_1} x^{i_2} \cdots x^{i_m}$$

be a polynomial in m *variables* x_i, *of degrees* N_i *in* x_i $(i = 1, 2, \cdots, m)$, *with rational integral coefficients satisfying the inequality* $|a_{i_1 i_2 \cdots i_m}| \leq h$. *Let* $\alpha_i \in \mathfrak{A}(n_i, h_i, s_i)$ $(i = 1, 2, \cdots, m)$ *be* m *algebraic numbers for which*

$$A(\alpha_1, \alpha_2, \cdots, \alpha_m) \neq 0;$$

and let q *be the degree of the field* $R(\alpha_1, \alpha_2, \cdots, \alpha_m)$ *over the field* R *of rational numbers. Then*

$$(2) \qquad |A(\alpha_1, \alpha_2, \cdots, \alpha_m)| \geq (8^{N_1+N_2+\cdots+N_m} h h_1^{N_1/n_1} h_2^{N_2/n_2} \cdots h_m^{N_m/n_m})^{-q}.$$

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