ON THE MAXIMAL DOMAIN OF A "MONOTONE" FUNCTION

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1. INTRODUCTION

Let $\mathfrak{X}$ be a Hilbert-space. In $\mathfrak{X} \times \mathfrak{X}$, we define the M-relation by

$$(x_1, y_1)\, M\, (x_2, y_2) \qquad \text{provided } \Re \langle x_1 - x_2, y_1 - y_2 \rangle \geq 0 .$$

(The symbol $\Re$ may be dropped if the scalars are real.) This relation has been studied in previous papers (for example [2], [3]). In harmony with these papers, we shall say that a set $E \subset \mathfrak{X} \times \mathfrak{X}$ is *totally-M-related* provided $(x_1, y_1), (x_2, y_2) \in E$ implies $(x_1, y_1)\, M\, (x_2, y_2)$. We shall say that a map F from a subset of $\mathfrak{X}$ into $\mathfrak{X}$ is *monotone* provided its graph in $\mathfrak{X} \times \mathfrak{X}$ is totally-M-related.

A *real linear subspace* is a subset $\mathfrak{X}_0 \subset \mathfrak{X}$ such that β_1, β_2 real and $x_1, x_2 \in \mathfrak{X}_0$ imply $\beta_1 x_1 + \beta_2 x_2 \in \mathfrak{X}_0$. A *real affine variety* is a translate of a real linear subspace. We call a set $Q \subset \mathfrak{X}$ *almost-convex* provided it contains the interior of its convex hull K(Q), where the "interior" is taken relative to the smallest real affine variety containing Q (or equivalently, K(Q)).

2. THE THEOREM

THEOREM. *Let $\mathfrak{X}$ be a finite-dimensional Hilbert-space, with real or complex scalars, and let $E \subset \mathfrak{X} \times \mathfrak{X}$ be a maximal totally-M-related set. Let P be the projection $P(x, y) = x$. Then P(E) is an almost-convex set.*

Proof. Our object is to show that int $K[P(E)] \subset P(E)$. For the moment, we restrict attention to the case where the scalars of $\mathfrak{X}$ are real. Let $x_0 \in \text{int } K[P(E)]$. Without loss of generality, we shall assume that x_0 is the zero-vector θ; for if this does not hold, the translation $x \to x - x_0$ (leaving the y's fixed) will carry E into a new maximal totally-M-related set E', and x_0 into θ, and so forth. Thus the "affine variety" of the theorem becomes "linear subspace."

Furthermore, we lose no generality by assuming that the "interior" is taken relative to the space $\mathfrak{X}$. Suppose that $\mathfrak{X}_0$, the subspace spanned by K[P(E)], is of dimension less than that of $\mathfrak{X}$, and let $\mathfrak{X}_0^{\perp}$ be the orthogonal complement of $\mathfrak{X}_0$. Then each vector y can be resolved as $y = y^0 + y^1$, where $y^0 \in \mathfrak{X}_0$ and $y^1 \in \mathfrak{X}_0^{\perp}$, and it is easily seen that $(x, y) \in E$ if and only if $(x, y^0) \in E$, and that the image E_0 of E under the map $(x, y) \to (x, y^0)$ is a maximal totally-M-related set in $\mathfrak{X}_0 \times \mathfrak{X}_0$ such that $P(E_0) = P(E)$.

With these assumptions, we proceed. Let S be a sphere with center θ such that $S \subset K[P(E)]$. It is easy to find a finite set F of vectors of S which generate $\mathfrak{X}$ (considered as a convex cone). Each vector of F, in turn, is a finite linear combination, with positive coefficients, of vectors of P(E), so that we can find a finite set $x_1, \cdots, x_m$ of vectors of P(E) which generate $\mathfrak{X}$.

Consider now the polyhedral convex set

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