# ON METRIC PROPERTIES OF COMPLEX POLYNOMIALS

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Let

$$f(z) = \prod_{\nu=1}^{n} (z - z_\nu) = z^n + \cdots .$$

This paper deals with metric properties of the lemniscate domain

$$E = \{ |f(z)| \leq 1 \} .$$

It will give (at least partial) answers to some problems raised by Erdös, Herzog and Piranian [2]. Also, some metric properties of continua of capacity 1 will be derived.

Section 1 treats the diameters of the components of E. After some counter-examples, a lower bound for the largest diameter will be given, for the case where $|z_\nu| \leq r \leq 1$.

In Section 2 it will be proved that E contains a disk of radius const $\cdot\, n^{-4}$, if $z_\nu \in [-2, +2]$.

In Section 3 it will, for instance, be shown that $d \leq 4 \cdot 2^{-1/n}$ and $\Lambda < 74n^2$, where d is the measure of the projection of E onto the real axis and $\Lambda$ is the perimeter of E.

Section 4 deals first with some necessary or sufficient conditions for the connectedness of E, and then with some consequences of connectedness.

The last section is concerned with the convexity of E and two related problems.

## 1. THE DIAMETERS OF THE COMPONENTS OF E

There is a close connection between lemniscate domains E and compact sets F with cap F = 1. Here cap F denotes the (logarithmic) capacity of F, also called the transfinite diameter of F. Every lemniscate domain $E = \{ |f(z)| \leq 1\}$ generated by $f(z) = z^n + \cdots$ has capacity 1 [4], and conversely the following approximation theorem holds [5]:

Let F be a closed bounded set with cap F = 1. Given any $\varepsilon > 0$ and $\eta > 0$, there exists a $\rho$ $(1 < \rho < 1 + \eta)$ and a polynomial $f(z) = z^n + \cdots$ such that the lemniscate $\{ |f(z)| = \rho^n\}$ contains F in its interior and is contained in an $\varepsilon$-neighborhood of F.

We shall now apply the approximation theorem to some problems of Erdös, Herzog and Piranian [2]. Let E have the components $E_j$, of diameters $d_j$. Then Problem 8 asks whether

$$\sum \max(0, d_j - 1)$$

Received November 26, 1960.