CONNECTED SETS OF WADA

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1. INTRODUCTION

It is well known that, in the closure of a connected domain D of the euclidean plane E_2, the familiar boring process of Wada [13, pp. 60-62] always gives an indecomposable continuum. Wilder has shown that this is not true in euclidean space E_m ($m \geq 3$); for in [11] he gives constructions which lead to locally connected continua in E_3. We are interested in various types of the Wada tunneling process, and especially in connected towers obtained by using this process a finite or infinite number of times; in fact, our interest was first aroused by the observation that the intersection of some towers of connected Wada domains are indecomposable connected sets with composant properties similar to those of an indecomposable continuum. Various modifications of the Wada tunneling give many types of sets; some of these do not seem now to be characterizable with words so as to give results of generality and interest. However, the Wada construction does lead to some of the more peculiar sets.

Our imbedding space is E_m ($m \geq 3$) or the Hilbert cube I_ω. We obtain indecomposable continua by several methods, all of which depend on modifications of the shielding that Hunter and I used in [2]. The set of perhaps the greatest interest (it is of a new type) we call a *connected set* M *with a set* Z *of indecomposability;* it is a generalization of an indecomposable continuum, since M is indecomposable when it is closed and $\overline{Z} = M$. In general, these new sets, obtained by modifying a netting type of construction used by Wilder in [11], have some properties of composants similar to those of an indecomposable continuum.

All the ideas used here, and most of the proofs, are simple. However it is very difficult to keep this simplicity from being hidden by the complexity of the notation needed for the Wada constructions. Whenever possible, an attempt is made to present an intuitive way of looking at these; this loses something in exactness, but may help to show the simplicity, when accompanied by indications of the more exact construction. The idea of shielding, used in [2], and that of a basic connexe densely extendable over a connected domain, used in [9], are devices whose purpose is to help the intuition; these are both used below. In other places one may find the figures and proofs given by Wilder in [11, pp. 276-278, 290-291] of help. Some of our methods are used, in part, in [2], [7] and [9]; these references may be helpful.

Below, D always denotes a connected domain of our space; in D we construct, by any one of various modifications of the Wada tunneling process, a domain D^1, which may or may not be connected but whose closure $\overline{D}^1$ is connected and is equal to $\overline{D}$. Thus we have a *descending tower* ($D \supset D^1 \supset D^2 \supset \cdots$) of Wada domains; since a tower suggests the upward direction, we put the index h of the h^{th} stage of the tower construction into the upper position: D^h. We denote the boundary of D^h by $F(D^h)$, and we observe that these boundaries give an *ascending tower*

Received August 5, 1958, and in revised forms August 1, 1959 and October 1, 1960.

Presented to the American Mathematical Society January 28 and November 28, 1958. Supported in part by a grant from the National Science Foundation.